A

SHORT TREATISE

ON THE

PRINCIPLES

OF THE

DIFFERENTIAL

AND

INTEGRAL

CALCULUS.

PART II.

DESIGNED FOR

THE USE OF STUDENTS IN THE UNIVERSITY.

BY THE

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PREFACE.

THE ensuing volume appears as the continuation and conclusion of an elementary course of analytical mathematics, which the author has been for some time past engaged in bringing out in the form of successive treatises, each in some measure distinct, yet all connected. The objects in view in such an undertaking have been sufficiently described in the prefatory notices prefixed to former volumes. In the present part nothing more is aimed at than the completion of the same design. The same principles have been adopted, and the same rules adhered to, by the compiler, for his guidance in the selection of materials and the mode of employing and presenting them, as those followed in the former portions of his work. It is his endeavour all along to compress into a small compass all that may be considered of most essential use to a learner; and to explain every thing which is introduced by the utmost fulness of detail.

The former part of the Short Treatise on the Differential and Integral Calculus contains its

elementary principles, with a very few applications: the present volume will be found to comprise more extended instances of such applications; together with a further discussion of some of the principles of the science, and of one or two branches of it not before entered upon; such discussion, however, being in all cases confined within the limits which the nature of a treatise strictly elementary should impose. The various applications may of course be followed up to just that extent which may suit the particular views of the student, after he has acquired a fundamental knowledge of the science from the former part.

The author feels it particularly incumbent on him to apologize to his readers for the detached and unconnected form in which certain parts of his design have been brought out. But the whole being intended rather as successive parts of one work than as distinct treatises, and as they are each necessary to the illustration of the rest, he has allowed himself to be guided rather by circumstances than by the proper connexion of the subject, as to the order in which different portions have appeared. But he trusts that the proper arrangement in which they follow one another will be so readily understood, that their being thus out of place will not occasion any perplexity to the student.

In particular, the articles on Vanishing Fractions, and on Integration, prefixed to the volume on "the Application of the Calculus to "the Geometry of Curves," may be easily transferred to their proper places in the system of the Calculus; as also the general principles of Maxima and Minima of two Variables in the same volume. The short sketch of Differential Equations, also, given at the end of the same volume, properly belongs to the present, and the section on the subject given here follows in immediate connexion with it. Of this section the author willingly takes this opportunity of saying that it will be found to be but an imperfect account of part of the subject which, in its full extent, is one at the same time of considerable difficulty, and of the highest interest: this section is almost wholly an abridgement of the corresponding portion of the treatises of Lacroix and Boucharlat; and the student whose views reach to following it up, will find due reference to fuller sources of information.

Throughout the whole work the details of the operations are always fully stated: and in doing so the author has generally adhered to the most direct and natural mode of deduction: better methods and more ingenious artifices will doubtless occur to the more practised analyst: but the investigations here given, it is hoped,

will be found at all events such as to answer the purpose intended, that of giving the requisite assistance to the learner, and leading him on by progressive steps to a thorough acquaintance with the principles of the science.

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THE PRINCIPLES

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PART II.

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CALCULUS.

DIFFERENTIAL CALCULUS.

IN the former part of this work a general elementary view was given of the principles of the Differential Calculus, and some of its chief applications. We shall here proceed to illustrate those principles somewhat more fully by a selection of examples; to supply additional instances of the applications of the Calculus to some important points; and to carry on the course of the investigation to certain parts of the subject which are essential to a more extended view of analysis.

DIFFERENTIATION.

IN the following section it is designed to give a few such examples of differentiation as present any thing remarkable either in the operation or in the result. These, with occasional remarks, may suffice to illustrate the nature of the process, and to afford a guide to the student in similar investigations.

ALGEBRAIC FUNCTIONS.

(1.) The differentiation of a square root often occurs; it may be useful to bear in mind the form which it takes:

$$\text{Let } u = x^{\frac{1}{2}}$$

$$\text{thence, } du = \tfrac{1}{2}x^{-\frac{1}{2}}dx = \frac{dx}{2\sqrt{x}}$$

$$\text{In like manner, } d\sqrt{1+x^2} = \frac{xdx}{\sqrt{1+x^2}}$$

(2.) To differentiate the reciprocal of the square root, or,

$$u = \frac{1}{\sqrt{x}}$$

$$\text{We have } du = \frac{-d\sqrt{x}}{x};$$

which, by substituting the value of the numerator from the last example,

$$= \frac{-dx}{2x^{\frac{3}{2}}}.$$

[3.] Let $u = \dfrac{x}{1+x}$,

$$\text{we have } du = \frac{dx\ (1+x) - xdx}{(1+x)^2}.$$

In the numerator two terms destroy each other, and there results,

$$d\left(\frac{x}{1+x}\right) = \frac{dx}{(1+x)^2}$$

(4.) In these and other examples a quantity involving x is given equal to u, and in this state is called an *explicit* function of x. But it is often required to differentiate an equation containing terms in which u and x are combined, which is called an *implicit* function of x. For example, let it be required to differentiate

$$u^2-2u\sqrt{1+x^2}+x^2=0;$$

here we have,

$$2udu-2\sqrt{1+x^2}\,du-\frac{2uxdx}{\sqrt{1+x^2}}+2xdx=0;$$

$$\text{Or } (u-\sqrt{1+x^2})\,du=\left(\frac{ux}{\sqrt{1+x^2}}-x\right)dx;$$

which is easily reduced to

$$du=\frac{x(u-\sqrt{1+x^2})dx}{\sqrt{1+x^2}\,(u-\sqrt{1+x^2})}=\frac{xdx}{\sqrt{1+x^2}}.$$

The student will doubtless remark, that this result is the same as that found by the differentiation of $u=\sqrt{1+x^2}$. [Ex. 1.] This will be easily seen to arise from the circumstance that the differential of $\sqrt{1+x^2}$ is the same as that of $\sqrt{1+x^2}+1$; and this last will be found to be the value of u deduced by solving the quadratic equation above given, or deriving from the implicit an explicit function of x.

(5.) It is sometimes required to differentiate continued roots or fractions, or series, in which cases the algebraical value must first be obtained, and then the expression differentiated. For example, let it be required to differentiate

$$u=\sqrt{x+\sqrt{x+\sqrt{+\sqrt{x+,\ \&c.}}}}$$

We first express its value by taking

$$u^2 = x + \sqrt{x + \sqrt{x +, \&c.}}$$

$$= x + u$$

$$\therefore u^2 - u = x.$$

And solving this quadratic equation we have,

$$u^2 - u + \tfrac{1}{4} = x + \tfrac{1}{4};$$

$$\text{whence, } u - \tfrac{1}{2} = \pm\sqrt{x + \tfrac{1}{4}}.$$

And this value being differentiated gives

$$du = \pm\frac{dx}{2\sqrt{x + \frac{1}{4}}}.$$

(6.) In a similar way, if it be required to differentiate

$$u = \cfrac{x^2}{1 + \cfrac{x^2}{1 + \cfrac{x^2}{1 + \&c. \text{ ad inf.}}}}$$

We must first take $u = \dfrac{x^2}{1+u} \therefore u^2 + u = x^2$;

$$\text{whence } u + \tfrac{1}{2} = \pm\sqrt{x^2 + \tfrac{1}{4}};$$

and this being differentiated gives

$$du = \pm\frac{xdx}{\sqrt{x^2 + \frac{1}{4}}}.$$

(7.) To differentiate the function

$$u = a + ax + ax^2 + \ldots \text{ to } n \text{ terms.}$$

These terms forming a geometrical series, we sum them by the well known formula, and finding,

$$u = a\frac{(x^n - 1)}{x - 1}$$

we then take

$$du = a\left(\frac{(x-1)\ d(x^n-1) - (x^n-1)\ d(x-1)}{(x-1)^2}\right)$$

$$= a\left(\frac{(x-1)\ nx^{n-1}\ dx - (x^n-1)\ dx}{(x-1)^2}\right)$$

$$= adx\left(\frac{nx^n - nx^{n-1} - x^n + 1}{(x-1)^2}\right)$$

$$= adx\left(\frac{(n-1)\ x^n - nx^{n-1} + 1}{(x-1)^2}\right)$$

TRANSCENDENTAL FUNCTIONS.

(8.) To differentiate $u = \log.\sqrt{\frac{1+\sin.\ x}{1-\sin.\ x}}$.

Writing $\sin.\ x = z$, we have

$$u = \tfrac{1}{2}\Big[\log.\ (1+z) - \log.\ (1-z)\Big],$$

and thence $du = \frac{1}{2}\left(\frac{dz}{1+z} - \frac{-dz}{1-z}\right)$.

And reducing to a common denominator

$$du = \tfrac{1}{2}\left(\frac{2}{(1+z)(1-z)}\right)dz = \frac{dz}{1-z^2};$$

but $dz = d\ \sin.\ x = dx\ \cos.\ x$, and $1 - z^2 = \cos.^2 x$.
Hence replacing these values,

$$du = \frac{dx\ \cos.\ x}{\cos.\ ^2 x} = \frac{dx}{\cos.\ x}.$$

By a precisely similar process we find the differential of $u = \log.\sqrt{\frac{1-\cos.\ x}{1+\cos.\ x}}$ to be $du = \frac{dx}{\sin.\ x}$.

(9.) To differentiate $u = \sin.^{-1}\left(\frac{x}{\sqrt{1+x^2}}\right)$,

writing $\frac{x}{\sqrt{1+x^2}} = v$, we have

$$dv = \frac{\sqrt{1+x^2}dx - x\frac{xdx}{\sqrt{1+x}}}{1+x^2} = \frac{dx\left(1-\frac{x^2}{1+x^2}\right)}{1+x^2} = \frac{dx\cos. u}{1+x^2}$$

But $dv = du \cos. u$. Hence $du = \frac{dx}{1+x^2}$.

(10.) To differentiate $u = \sin.^{-1}\left(\frac{1-x^2}{1+x^2}\right)$.

Writing $\frac{1-x^2}{1+x^2} = v$, we find

$$dv = \frac{(1+x^2)(-2xdx) - (1-x^2)2xdx}{(1+x^2)^2} = \frac{-4xdx}{(1+x^2)^2}.$$

But $dv = du \cos. u = du\sqrt{1-v^2} \therefore du = \frac{dv}{\sqrt{1-v^2}}$.

And $\sqrt{1-v^2} = \sqrt{\frac{(1+x^2)^2-(1-x^2)^2}{(1+x^2)^2}} = \sqrt{\frac{4x^2}{(1+x^2)^2}} = \frac{2x}{1+x^2}$

Hence $du = \frac{-4xdx}{(1+x^2)^2} \cdot \frac{1+x^2}{2x} = \frac{-2dx}{1+x^2}$.

(11.) The rule for differentiating $y = a^x$ may be extended to more complex cases.

Let $y = a^{b^x}$, writing $b^x = u$ it becomes $y = a^u$; therefore

$$\frac{dy}{du} = a^u \log. a = a^{b^x} \log. a$$

$$\frac{du}{dx} = b^x \log. b.$$

Wherefore $\frac{dy}{dx} = a^{b^x} b^x \log. a. \log. b.$

(12.) Let $y = z^v$. Taking the logarithms

$$\log. y = v \log. z$$

$$\therefore d \log. y = v. d. \log. z + \log. z. dv;$$

$$\text{or } \frac{dy}{y} = v\frac{dz}{z} + \log. z\, dv$$

$$\therefore dy = y\left(\frac{vdz}{z} + \log. zdv\right).$$

(13.) Hence we may deduce the differential of $y = z^{t^u}$; for let $t^u = v$, $\therefore y = z^v$.

Then by the preceding form, we have in these two cases,

$$dy = z^v\left(v\,\frac{dz}{z} + zdv\right)$$

$$dv = t^u\left(u\,\frac{dt}{t} + \log. tdu\right).$$

Substituting the values of v and dv in that of dy,

$$dy = z^{t^u}\left[t^u\,\frac{dz}{z} + \log. zt^u\left(u\,\frac{dt}{t} + \log. tdu\right)\right]$$

$$= z^{t^u}.\, t^u\left(\frac{dz}{z} + u \log. z\,\frac{dt}{t} + \log. z. \log. t. du\right).$$

(14.) To differentiate $u = \log. \sqrt{\frac{1+x}{1-x}} = \frac{1}{2} \log \left(\frac{1+x}{1-x}\right)$,

we have $du = \frac{1}{2}\left\{\dfrac{d\left(\dfrac{1+x}{1-x}\right)}{\dfrac{1+x}{1-x}}\right\}$

$$= \frac{1}{2}\left\{\frac{(1-x)dx + (1+x)dx)}{(1-x)^2}\left(\frac{1-x}{1+x}\right)\right\}$$

$$= \frac{1}{2}\left\{\frac{2dx}{(1-x)(1+x)}\right\} = \frac{dx}{1-x^2}.$$

(15.) The following cases should be viewed in connexion :

Let $u = \log. [x + \sqrt{x^2 - 1}]$

$$du = \left\{\frac{dx + \dfrac{xdx}{\sqrt{x^2-1}}}{x + \sqrt{x^2-1}}\right\}$$

$$= \left\{\frac{dx\,(\sqrt{x^2-1} + x)}{\sqrt{x^2-1}\,(x + \sqrt{x^2-1})}\right\} = \frac{dx}{\sqrt{x^2-1}}$$

(16.) Next let $u = \log. [x\sqrt{-1} + \sqrt{x^2-1}\sqrt{-1}]$, observing that $\sqrt{x^2-1}\sqrt{-1} = \sqrt{1-x^2}$, we have

$$du = \left\{\frac{dx\sqrt{-1} - \dfrac{xdx}{\sqrt{1-x^2}}}{x\sqrt{-1} + \sqrt{1-x^2}}\right\}$$

$$= \frac{dx(\sqrt{-1}\sqrt{1-x^2} - x)}{\sqrt{1-x^2}(x\sqrt{-1} + \sqrt{1-x^2})}$$

or multiplying both numerator and denominator by $\sqrt{-1}$,

$$= \frac{dx\sqrt{-1}\,(\sqrt{-1}\sqrt{1-x^2} - x)}{\sqrt{1-x^2}(-x + \sqrt{-1}\sqrt{1-x^2})} = \frac{dx\sqrt{-1}}{\sqrt{1-x^2}},$$

which again multiplied by $\sqrt{-1}$, gives

$$du = \frac{-dx}{\sqrt{x^2-1}}.$$

(17.) In this last case we may observe that,

$$d\log.(x\sqrt{-1} + \sqrt{1-x^2}) = \frac{\sqrt{-1}\,dx}{\sqrt{1-x^2}}$$

may be written

$$d\,.\,\frac{1}{\sqrt{-1}}\log.\ (x\sqrt{-1} + \sqrt{1-x^2}) = \frac{dx}{\sqrt{1-x^2}}.$$

(18.) And in the same manner we should find

$$d\,.\,\frac{1}{\sqrt{-1}}\log.\ (x\sqrt{-1} + \sqrt{1+x^2}) = \frac{dx}{\sqrt{1+x^2}}.$$

(19.) In precisely the same way as in example (15), we find

$$d\,.\log.\ (\sqrt{1+x^2}+x) = \frac{dx}{\sqrt{1+x^2}}.$$

(20.) And again in the same manner,

$$d\,.\log.\ (\sqrt{1+x^2}-x) = dx\left\{\frac{x-\sqrt{1+x^2}}{(\sqrt{1+x^2}-x)\sqrt{1+x^2}}\right\}$$

$$= \frac{-(\sqrt{1+x^2}-x)}{(\sqrt{1+x^2}-x)\sqrt{1+x^2}} = \frac{-dx}{\sqrt{1+x^2}}.$$

(21.) Hence we easily differentiate

$$u = \log.\ \left\{\frac{\sqrt{1+x^2}+x}{\sqrt{1+x^2}-x}\right\}^{\frac{1}{2}}.$$

For we must take

$$du = \tfrac{1}{2}\left(d\log.\ (\sqrt{1+x^2}+x) - d\log.\ (\sqrt{1+x^2}-x)\right),$$

and writing for these their values above found,

$$du = \tfrac{1}{2}\frac{2\,dx}{\sqrt{1+x^2}} = \frac{dx}{\sqrt{1+x^2}}.$$

(22.) The two following forms, though similar in appearance, require different methods of differentiation.

$$\text{Let } u = \log.\left\{\frac{\sqrt{1+x}+\sqrt{1-x}}{\sqrt{1+x}-\sqrt{1-x}}\right\};$$

or writing the fraction $=\dfrac{y}{z}$, $u = \log.\, y - \log.\, z$.

$$\text{And } du = \frac{dy}{y} - \frac{dz}{z}$$

$$dy = \frac{dx}{2\sqrt{1+x}} - \frac{dx}{2\sqrt{1-x}} = \frac{dx(2\sqrt{1-x}-2\sqrt{1+x})}{4\sqrt{1-x^2}}$$

$$= \frac{-dx\,z}{2\sqrt{1-x^2}}$$

$$dz = \frac{dx}{2\sqrt{1+x}} + \frac{dx}{2\sqrt{1-x}} = \frac{dx(2\sqrt{1-x}+2\sqrt{1+x})}{4\sqrt{1-x^2}}$$

$$= \frac{dx\,y}{2\sqrt{1-x^2}}$$

$$\therefore du = dx\left\{\frac{-z}{2y\sqrt{1-x^2}} - \frac{y}{2z\sqrt{1-x^2}}\right\};$$

or reducing to a common denominator

$$du = dx\left\{\frac{-(z^2+y^2)}{2yz\sqrt{1-x^2}}\right\}.$$

But on squaring the numerator and denominator of the original fraction, and adding, it is evident that the second terms of the two squares destroy each other, and we have

$$y^2 + z^2 = (1+x) + (1-x) + (1+x) + (1-x) = 4.$$

Also the product $yz = (1+x) - (1-x) = 2x$.

Hence $du \dfrac{-dx}{x\sqrt{1-x^2}}$.

(23.) To differentiate

$$u = \log.\left(\frac{\sqrt{1+x}-\sqrt{1-x}}{\sqrt{1+x}+\sqrt{1-x}}\right).$$

We must first observe, that the quantity within the brackets is reducible to a simpler form; for by multiplying both numerator and denominator by the quantity

$$\sqrt{1+x}-\sqrt{1-x},$$

there results, $\dfrac{(\sqrt{1+x}-\sqrt{1-x})^2}{(1+x)-(1-x)}$;

which becomes by expanding and reducing

$$\left(\frac{1-\sqrt{1-x^2}}{x}\right).$$

To differentiate the logarithm of this quantity we take

$$du = d\,(\log.(1-\sqrt{1+x^2})-\log.\,x)$$

$$= \left\{\frac{\dfrac{xdx}{\sqrt{1-x^2}}}{1-\sqrt{1-x^2}} - \frac{dx}{x}\right\}$$

$$= dx\left(\frac{x^2-(1-\sqrt{1-x^2})\sqrt{1-x^2}}{\sqrt{1-x^2}\,(1-\sqrt{1-x^2})x}\right)$$

$$= dx\left(\frac{x^2-\sqrt{1-x^2}+1-x^2 = 1-\sqrt{1-x^2}}{(\sqrt{1-x^2})\,(1-\sqrt{1-x^2})x}\right)$$

$$= dx \frac{1}{x\sqrt{1-x^2}}.$$

(24.) In like manner if we differentiate

$$u = \log.\left(\frac{1+\sqrt{1-x^2}}{x}\right)$$

we shall find by the same process

$$du = -\frac{dx}{x\sqrt{1-x^2}}.$$

(24.) The following method is often useful for facilitating the process of differentiation.

If we have two equations

$$y = fu$$
$$u = \phi x.$$

From these we find separately the values of $\frac{dy}{du}$ and $\frac{du}{dx}$, and multiplying them together, we have the value of $\frac{dy}{dx}$.

For example—let there be given

$$y = 3u^2 \text{ and } u = x^3 + ax^2,$$

we have, $\frac{dy}{du} = 6u \qquad \frac{du}{dx} = 3x^2 + 2ax.$

Hence $\frac{dy}{dx} = 6u\ (3x^2 + 2ax)$

$$= 6\ (x^3 + ax^2)\ (3x^2 + 2ax).$$

In complex expressions this process sometimes affords great simplification.

SUCCESSIVE DIFFERENTIATION.

(1.) The successive differentials of the function $\frac{1}{x}$ are somewhat remarkable: they will be found as follows;

$$u = \frac{1}{x}$$

$$\frac{du}{dx} = -\frac{1}{x^2}$$

$$\frac{d^2u}{dx^2} = \frac{2.1}{x^3}$$

$$\frac{d^3u}{dx^3} = -\frac{3.2.1}{x^4}$$

. . . .

$$\frac{d^nu}{dx^n} = (-1)^n \frac{n.(n-1)\ldots.3.2.1}{x^{n+1}}.$$

(2.) Let $u = \frac{1}{(1-x)^2}$.

Then it will be readily found that we have

$$\frac{du}{dx} = \frac{-2(1-x)(-1)}{(1-x)^4} = \frac{2}{(1-x)^3};$$

and proceeding in the same manner, we shall evidently arrive at

$$\frac{d^nu}{dx^n} = \frac{1.2.3.4\ldots(n+1)}{(1-x)^{n+2}}.$$

(3.) From the differential coefficients of this function we have those of

$$u = \frac{1+x}{1-x};$$

for we obtain directly

$$\frac{du}{dx}=\frac{2}{(1-x)^2}.$$

Hence the successive differential coefficients will be those of the function $\frac{1}{(1-x)^2}$, multiplied by 2, of the order next below: thus we shall have in this case

$$\frac{d^n u}{dx^n}=\frac{2.2.3\ldots\ldots n}{(1-x)^{n+1}}.$$

The determination of the n^{th} differential coefficient of a given function without going through the previous differentiations, is a point of considerable importance in complex developements: it gives what is called the "general term" of the series, which expresses its law: and the subject has accordingly engaged much of the attention of analysts, and formulæ have been given for various forms of functions; but as they are necessarily of a complex nature, we shall not here pursue the subject. The reader will find several such forms stated in Peacock's Examples, p. 11, and the full discussion of them in Lacroix's large treatise, ch. 1. art. 37.

(4.) In the developement of a^x (Diff. Calc. p. 33.) the student will doubtless have remarked that the successive differential coefficients involve the original function multiplied by the successive powers of A. In the case therefore of the Napierian logarithm where $A=1$, or in the developement of e^x, we have the remarkable property that *all the differential coefficients are equal to each other and to the original function.*

(5.) The same remark may be made also with respect to the differential coefficients of sin. x and cos. x: (Diff. Calc. p. 32.) Those of even orders being equal to the original function but with signs alternately negative and positive

ELIMINATION OF CONSTANTS, &c. BY DIFFERENTIATION.

(1.) It is obvious that the process of differentiation will enable us to get rid of any *constant* quantity which is *not combined* with a variable in a given expression; but we can by means of the given equation and the resulting differential equation also get rid of a *constant* which enters as a *coefficient* of a variable.

For example, if we had the equation

$$y^2 = ax + b;$$

Differentiating it, we find

$$2ydy = adx;$$

and by means of these two expressions we can eliminate a; since we have

$$\frac{y^2}{x} - \frac{b}{x} = a = \frac{2ydy}{dx};$$

which gives the differential equation

$$y^2dx = 2yxdy + bdx,$$

a result which is independent of a.

Dividing by dx, this gives

$$y^2 = 2yx\frac{dy}{dx} + b;$$

and differentiating again, we have

$$2y\frac{dy}{dx} = 2yx\frac{d^2y}{dx} + 2y\frac{dy}{dx}dx + 2x\frac{dy^2}{dx},$$

a result which is independent both of a and b.

(2.) If the constant be above the first degree, the resulting equation will contain higher powers of dy and dx than the first. For example, let

$$y^2 - 2ay + x^2 = a^2.$$

Differentiation gives us

$$ydy - ady + xdx = 0.$$

Whence we have

$$a = \frac{ydy + xdx}{dy}.$$

Substituting this in the given equation, it becomes

$$y^2 - 2y\,\frac{(ydy + xdx)}{dy} + x^2 = \frac{(ydy + xdx)^2}{dy^2}.$$

Or,

$$y^2 - 2y^2 - 2yx\frac{dx}{dy} + x^2 = y^2 + 2yx\frac{dx}{dy} + x^2\frac{dx^2}{dy^2};$$

which, transposing and multiplying by $\frac{dy^2}{dx^2}$, becomes

$$(x^2 - 2y^2)\frac{dy^2}{dx^2} - 4xy\frac{dy}{dx} - x^2 = 0.$$

An equation shewing the relation between the variable x, its function y, and the differential coefficient, *independent of the value of a.*

We might also have obtained the same result by solving the original equation for a, which would give

$$a = -y \pm \sqrt{2y^2 + x^2}.$$

And by differentiating this expression, we have

$$0 = -dy \pm \frac{2ydy + xdx}{\sqrt{2y^2 + x^2}}.$$

And hence, multiplying by the denominator, transposing and squaring, we have

$$(2ydy + xdx)^2 = dy^2(2y^2 + x^2).$$

Expanding, changing all the signs, and dividing by dx^2, we finally obtain

$$(x^2 - 2y^2)\frac{dy^2}{dx^2} - 4xy\frac{dy}{dx} - x^2 = 0,$$

which is the same equation as that before obtained.

(3.) Any number of constants may be made to disappear by differentiating as many times as there are constants in the equation. For example, let there be given

$$y^2 = m\,(a^2 - x^2)\,;$$

we have $\quad ydy = -mxdx$

or $y\dfrac{dy}{dx} = -mx.$

Differentiating again, we find

$$y\frac{d^2y}{dx} + \frac{dy^2}{dx} = -mdx\,;$$

and substituting for m its value derived from the former equation,

$$m = \frac{-ydy}{xdx},$$

this becomes

$$yx\frac{d^2y}{dx} + x\frac{dy^2}{dx} + ydy = 0.$$

an equation independent of the constants a and m.

(4.) By methods similar to the preceding we can make irrational functions disappear from equations; ob-

taining instead, equations involving differentials. Thus let there be given

$$(fx)^{\frac{p}{q}} = fy,$$

by differentiating we have

$$\frac{p}{q}(fx)^{\frac{p-1}{q}} d(fx) = d(fy),$$

$$\text{or, } \frac{p}{q}(fx)^{\frac{p}{q}} d(fx) = d(fy)(fx);$$

and substituting for $(fx)^{\frac{p}{q}}$ its value, we obtain

$$\frac{p}{q}(fy)\ d(fx) = (fx)\ d(fy),$$

an equation in which (fx) is freed from its fractional index.

The same result might have been obtained by taking the logarithms: by which means we should have

$$\frac{p}{q}\log.(fx) = \log.(fy).$$

Whence $\frac{p}{q}\frac{d(fx)}{fx} = \frac{d(fy)}{fy}$,

and $\frac{p}{q}(fy)\ d(fx) = (fx)\ d(fy)$

as before.

(5.) On the same principle we can remove transcendental terms from an equation: for example, let

$$u = \log.(a + bx).$$

Then $du = \frac{bdx}{a+bx}$, or $\frac{du}{dx} = \frac{b}{a+bx}$.

Again, let us take a circular function

$$u = \sin. x.$$

$$du = dx \cos. x = dx\sqrt{1 - x^2}$$

$$\cos. x = \frac{du}{dx}.$$

By differentiating a second time, we have

$$d^2u = d^2x \cos. x - dx^2 \sin. x;$$

and substituting the value of cos. x, this becomes

$$d^2u = d^2x \frac{du}{dx} - dx^2u.$$

Whence $dxd^2u = d^2xdu - dx^3u$.

MAXIMA AND MINIMA OF FUNCTIONS OF ONE VARIABLE.

We shall here give a few examples illustrative of the principle of the investigation of Maxima and Minima, to which some general remarks will be added.

We will commence with an application of the process to some very simple cases:

(1.) Let $y=b-cx^2$.

Here $\frac{dy}{dx}=-2cx \qquad \frac{d^2y}{dx^2}=-2c.$

There is only one value, viz. $x=0$, which can render $\frac{dy}{dx}=0$. And since the second differential coefficient is *negative*, this value, which gives $y=b$, belongs to a *maximum*.

Again, let $y=b+cx^2$

$$\frac{dy}{dx}=2cx, \qquad \frac{d^2y}{dx^2}=2c.$$

Here the value $x=0$, which gives $y=b$, since the second differential is *positive*, is a *minimum* value of the function.

But if the function had been

$$y=b\pm cx,$$

we should have had

$$\frac{dy}{dx}=\pm c, \qquad \frac{d^2y}{dx^2}=0,$$

and all the subsequent differential coefficients vanish also, or $x=0$ belongs neither to a maximum nor a minimum.

If again we had

$$y=b+cx^3$$

$$\frac{dy}{dx}=3.cx^2, \qquad \frac{dy^2}{dx^2}=2.3.c.x.$$

Here two values $+x$ and $-x$ give $\frac{dy}{dx}=0$; but they also make $\frac{d^2y}{dx^2}=0$. Also all the differential coefficients after the third are $=0$: hence these values give no maximum or minimum.

(2.) To find when $u=x^4-2x^2$ is a maximum or minimum.

$$\frac{du}{dx}=4x^3-4x$$

$$\frac{d^2u}{dx^2}=3.4.x^2-4.$$

For the maximum or minimum

$$4x^3-4x=0 \quad \therefore 4x(x-1)=0$$

$$\therefore \text{either } x=1, \text{ or } x=0.$$

If $x=1$ $\frac{d^2u}{dx^2}=12-4=8 \quad \therefore u=$ minimum.

If $x=0$ $\frac{d^2u}{dx^2}=-4 \quad \therefore u=$ a maximum.

(3.) Let $u\sqrt{a^2+x^2}-x$,

we have

$$\frac{du}{dx}=\frac{x}{\sqrt{a^2=x^2}}-1$$

$$\frac{d^2u}{dx^2}=\frac{\sqrt{a^2+x^2}-x\dfrac{x}{\sqrt{a^2+x^2}}}{a^2+x^2}=\frac{a^2}{(a^2+x^2)^{\frac{3}{2}}}.$$

The first differential coefficient can only become $=0$ when $x=\infty$: and the second is *positive* for all values of x. Hence $x=\infty$ belongs to a *minimum.*

(4.) To divide a quantity a into two parts x and $a-x$, such that the product $x^m(a-x)^n$ shall be a maximum.

Here therefore $u=x^m\ (a-x)^n$.

And $$\frac{du}{dx}=mx^{m-1}(a-x)^n-x^m n(a-x)^{n-1}$$

$$=\left(mx^{m-1}(a-x)-nx^m\right)(a-x)^{n-1}$$

$$=\left(ma-mx-nx\right)x^{m-1}\ (a-x)^{n-1}$$

$$=\left(ma-(m+n)x\right)x^{m-1}\ (a-x)^{n-1}.$$

Again, $$\frac{du^2}{dx^2}=\begin{cases}-(m+n)\,x^{m-1}(a-x)^{n-1}\\ +\left(ma-(m+n)x\right)(m-1)\,x^{m-2}(a-x)^{n-1}\\ -\left(ma-(m+n)x\right)x^{m-1}(n-1)\,(a-x)^{n-2}\end{cases}$$

$$\frac{du}{dx}=0 \text{ if } x=0,\ x=a, \text{ or, } x=\frac{ma}{m+n}.$$

The two first values make $\dfrac{d^2u}{dx^2}$ also vanish: but from the nature of the function it may be shewn that they belong to *minima* if m and n are *even* numbers: for each of the factors, after passing through the value 0, becomes negative, but the even powers being still posi-

tive, we have positive and increasing values for the function u. This might be shewn by referring to the successive differentials.

The third value substituted in the second differential coefficient, gives

$$\frac{d^2u}{dx^2} = -(m+n)\,\frac{ma^{m-1}}{(m+n)^{m-1}}\,\frac{[(m+n)\,a - ma]^{n-1}}{(m+n)^{n-1}}$$

$$= \frac{-m\;n^{n-1}\;a^{m+n-2}}{(m+n)^{m+n-3}}.$$

Or this value corresponds to a *maximum*.

(5.) To find the maximum or minimum values of the function,

$$u = \frac{x^2+1}{x} = x + \frac{1}{x}.$$

By differentiating we have

$$\frac{du}{dx} = 1 - \frac{1}{x^2},$$

$$\frac{d^2u}{dx^2} = \frac{2x}{x^4}.$$

To have the first differential $=0$, we must take

$$1 - \frac{1}{x^2} = 0, \text{ or } x^2 = 1 \quad \therefore x = \pm 1.$$

And it is evident that the second differential will have the same sign as x. Hence

$$x = +1 \text{ gives } u = +2 \text{ a minimum,}$$

$$\text{and } x = -1 \text{ gives } u = -2 \text{ a maximum.}$$

This example serves to remind the student of the entirely *relative* nature of maxima and minima; the *absolute* value being here the same in both cases.

We may also here make another observation, which will sometimes abridge the calculation of maxima and minima; viz. the same value which makes a function u a maximum, must make its reciprocal $\frac{1}{u}$ a minimum; and conversely. It may happen that the differentiation of a function may be much more complicated than that of its reciprocal. In which case we may choose that of the two which requires the shortest process, and thence infer the maximum and minimum of the other. Thus in the present case, if it were required to find the maximum or minimum of the reciprocal of the function we have been considering, or of

$$u_2 = \frac{x}{x^2+1},$$

the differentiation of this fraction would be more complex than that of the former: and we shall more easily obtain our result by means of those above deduced; from which we find that the value

$$x = +1 \text{ gives } u_2 = \frac{1}{2}, \text{ a maximum,}$$

and $\qquad x = -1$ gives $u_2 = -\frac{1}{2}$ a minimum.

(6.) The introduction of logarithms sometimes facilitates the investigation of maxima and minima, as will be best understood from the following example:

$$\text{let } u = \sqrt{x^2 - ax + b} \,.\, \sqrt[3]{m - x^3},$$

then when the function is a maximum or minimnm its logarithm is so likewise. We may therefore take

$$u_2 = \log.\, u = \frac{1}{2} \log.\, (x^2 - ax + b) + \frac{1}{3} \log.\, (m - x^3).$$

And differentiating

$$\frac{du_2}{dx}=\frac{1}{2}\,\frac{2x-a}{x^2-ax+b}-\frac{1}{3}\cdot\frac{3x^2}{m-x^3}.$$

Whence

$$\frac{d^2u_2}{dx^2}=\frac{1}{2}\frac{(x^2-ax+b)2-(2x-a)(2x-a)}{(x^2-ax+b)^2}-\frac{2(m-x^3)x+x^2.3x^2}{(m-x^3)^2}$$

$$=\frac{1}{2}\frac{2(x^2-ax+b)-(2x-a)^2}{(x^2-ax+b)^2}-\frac{2(m-x^3)x+3x^4}{(m-x^3)^2}$$

$\dfrac{du_2}{dx}=0$ if $x=0$, which makes $u=\sqrt{b}\cdot\sqrt[3]{m}$; and $x=0$ makes $\dfrac{d^2u_2}{dx^2}=\frac{1}{2}\dfrac{2b-2a^2}{b^2}$; which will be positive or negative according as b is greater or less than a^2, or we shall have a minimum or maximum accordingly.

(7.) The following method will sometimes shorten the process of a second differentiation for finding whether a given value of a function belongs to a maximum or a minimum. Let fx, ϕx, be functions of x, fx being such as to vanish when a particular value is given to x, but not ϕx. And let it be supposed that we have found

$$\frac{dy}{dx}=fx\,.\,\phi x:$$

differentiating again, we have

$$\frac{d^2y}{dx^2}=\frac{fx\,.\,d\,\phi x}{dx}+\frac{\phi x\,.\,d.fx}{dx}.$$

But when we assign to x the particular value supposed at which $fx=0$, this becomes

$$=\frac{\phi x\,.\,d.fx}{dx}.$$

That is, in order to obtain the value of $\dfrac{d^2y}{dx^2}$ for this

value of x, we have only to *differentiate the factor, which vanishes, and multiply by the other.* In this, however, it is supposed that $d.fx$ does not vanish also. If it does, we must recur to the usual method.

We will exemplify this process in the following case:

$$\text{Let } u = \frac{a+x}{x}\sqrt{b^2+x^2} = \left(\frac{a}{x}+1\right)\sqrt{b^2+x^2}$$

$$\frac{du}{dx} = \frac{a+x}{x} \cdot \frac{x}{\sqrt{b^2+x^2}} + \sqrt{b^2+x^2}\left(\frac{-a}{x^2}\right)$$

$$= \frac{ax^2+x^3-ab^2-ax^2}{x^2\sqrt{b^2+x^2}} = \frac{x^3-ab^2}{x^2\sqrt{b^2+x^2}}.$$

In order that this expression may be $=0$,

we must have $x^3 = ab^2$,

or $x = \sqrt[3]{ab^2}$.

To find whether this belongs to a maximum or a minimum, we shall have, by adopting the process just described,

$$fx = x^3 - ab^2 \qquad \phi x = \frac{1}{x^2\sqrt{b^2+x^2}};$$

and consequently for this value of x

$$\frac{d^2u}{dx^2} = \frac{3x^2}{x^2\sqrt{b^2+x^2}} = \frac{3}{\sqrt{b^2+x^2}}:$$

a *positive* result: or the value of x corresponds to a *minimum.*

(8.) If we have the function

$$u = 10x^6 - 12x^5 + 15x^4 - 20x^3 + 20;$$

then we find by differentiating

$$\frac{du}{dx}=60\ (x^5-x^4+x^3-x^2);$$

$$\text{and } \frac{d^2u}{dx^2}=60\ (5x^4-4x^3+3x^2-2x).$$

The first differential coefficient is easily resolved into its factors: the variable part of it may be written

$$x^2\ (x^3-x^2+x-1);$$

which is readily seen to be the product of

$$x^2\ (x-1)\ (x^2+1)$$

$$=x^2\ (x-1)\ (x+\sqrt{-1})\ (x-\sqrt{-1}):$$

which will become $=0$, if we have

$$\pm x=0,\ x=1,\ x=-\sqrt{-1},\ \text{or},\ x=+\sqrt{-1}.$$

The first gives $u=20$;

but it makes $\frac{d^2u}{dx^2}=0$, which is neither $+$ nor $-$, and it consequently gives neither a maximum nor a minimum.

The second value gives

$$u=10-12+15-20+20=13,$$

and makes $\frac{d^2u}{dx^2}=60\ (5-4+3-2)$, which is evidently positive, and therefore gives $u=13$, a minimum.

The third and fourth values substituted in $\frac{d^2u}{dx^2}$, since that expression contains odd powers of x, will give a result containing imaginary quantities, from which we can determine nothing as to maxima or minima.

(9.) We will here advance to some general considerations relative to the values of all algebraic functions which belong to maxima or minima.

If we have a function such that its first differential coefficient is resolvable into simple factors, with *unequal* values, or

$$\frac{du}{dx}=(x-a)\ (x-b)\ (x-c)\ (x-d)\ \&c.$$

whose roots are a, b, c, d, &c. which have their magnitudes in the order of the letters, then when x is made equal to any one of these roots, we may have a maximum or minimum. To find which it will be, we proceed to the second differential coefficient, which is,

$$\frac{d^2u}{dx^2}=\left.\begin{array}{l}[\,(x-b)\ (x-c)\ (x-d)\ \&c.\,]\\ +[\,(x-a)\ (x-c)\ (x-d)\ \&c.\,]\\ +[\,(x-a)\ (x-b)\ (x-d)\ \&c.\,]\\ +\ \&c.\end{array}\right\}$$

Now when x is made successively equal to each of the roots, we find that the *sign* of the whole will differ at each substitution: thus the supposition

$x=a$ gives $(a-b)(a-c)(a-d)\&c.=(+)(+)(+)\&c.=(+)$

$x=b\ldots(b-a)(b-c)(b-d)\ \&c.=(-)(+)(+)\ \&c.=(-)$

$x=c\ldots(c-a)(c-b)\ (c-d)\&c.=(-)(-)(+)\ \&c.=(+)$

&c.... &c. . . . . $=(-)$.

We thus see that the *real, unequal,* roots of the equation $\frac{du}{dx}=0$ substituted for x in the order of their magnitudes in the value of $\frac{d^2u}{dx^2}$, give results ultimately

positive and negative; and will consequently make u alternately a minimum and a maximum, beginning with the first. And if there are m real and unequal roots, there are m maximum or minimum values of the function.

When the equation has *equal* roots, the investigation is less simple; but we may explain it in the following manner:

First, let us suppose that the equation $\frac{du}{dx}=0$ has p equal roots, or that we have

$$\frac{du}{dx}=(x-a)^p.$$

Then we find by differentiating again,

$$\frac{d^2u}{dx^2}=p\,(x-a)^{p-1}.$$

And we shall not have a result which does not vanish when $x=a$, till we come to the differential of the order $(p+1)$, in which the index is $p-p=0$, giving

$$(x-a)^0=1\,;$$

$$\text{or, } \frac{d^{p+1}u}{dx^{p+1}}=p\,(p-1)\,\ldots\,3.2.1.\ (1).$$

If p be an even number, this differential is of an odd order, and therefore belongs to neither a maximum nor a minimum. If p be an odd number, this differential is of an even order; and its sign will evidently be positive, or it corresponds to a *minimum*. *Thus if the first differential coefficient has an odd number* (p) *of equal roots, the function admits of only one minimum value.*

Next let us take the case when there are two sets of

equal roots, or p roots equal to a, and q roots equal to b; that is, let

$$\frac{du}{dx}=(x-a)^p\,(x-b)^q.$$

Then differentiating again we find,

$$\frac{d^2u}{dx^2}=p(x-a)^{p-1}(x-b)^q+(x-a)^p\,q\,(x-b)^{q-1}$$

$$\frac{d^3u}{dx^3}=\begin{cases} p\cdot(p-1)(x-a)^{p-2}(x-b)^q+p)x-a)^{p-1}\,q(x-b)^{q-1} \\ +p(x-a)^{p-1}\,q(x-b)^{q-1}+(x-a)^p\,q\,(q-1)(x-b)^{q-2}. \end{cases}$$

And continuing in this way we must proceed to the differential coefficient of the $(p+1)$th order to have a result, one term of which will involve $(x-a)^0=1$, which will not vanish when we make $x=a$, but all the other terms will, since they involve powers of $(x-a)$ whose indices are greater than 0. This result will be of the form

$$\frac{d^{p+1}u}{dx^{p+1}}=p\,(p-1)\;\ldots\;3.2.1\;(1)\;(x-b)^q,$$

and when we substitute $x=a$ this becomes

$$=p\,(p-1)\;\ldots\;3.2.1\;(1)\,(a-b)^q.$$

And since by supposition $a>b$, this will be positive: and p being an odd number, the $(p+1)$th differential is of an *even* order, or the value belongs to a *minimum.*

In like manner the $(q+1)$th differential coefficient will give a result, one term of which involves $(x-b)^0$, and all the rest containing higher powers will vanish when $x=b$. The result will therefore be of the form

$$\frac{d^{q+1}u}{dx^{q+1}}=q\,(q-1)\;\ldots\;3.2.1\;(1)\;(x-a)^p,$$

which, on substituting $x=b$, becomes

$$=q\,(q-1)\ \ldots\ 3\,.\,2\,.\,1\ (1)\ (b-a)^p;$$

but since $b<a$, and p is an odd number, this result will be *negative:* or the value $x=b$ belongs to a *maximum.*

We might proceed exactly in the same way if we had a greater number of factors containing equal roots: and we conclude in general that there is *one value corresponding to each class of factors, which will belong to a maximum or a minimum,* according to the conditions above stated.

If the equation contain imaginary roots, the result of the substitution of these values gives an expression, involving imaginary terms, for $\frac{d^2u}{dx^2}$; from which we can infer neither a maximum nor a minimum.

(10.) To investigate the maxima or minima of the transcendental function,

$$u=x^x.$$

In order to get the differential coefficients we take

$$\log.\ u=x\log.\ x,$$

and thence,

$$d\,.\log.\ u=x\,.\,d\log.\ x+\log.\ x\,.\,dx;$$

$$\text{or, } \frac{du}{x^x}=x\frac{dx}{x}+\log.\ x\,.\,dx$$

$$\therefore\frac{du}{dx}=x^x\,(1+\log.\ x).$$

Let this be written $=v$; then to find the second differential,

$$\log.\ v=x\log.\ x+\log.\ [1+\log.\ x]$$

$$d\log.\ v=xd\log.\ x+dx\log.\ x+d\log.\ (1+\log.\ x)$$

$$\frac{dv}{v} = x\,\frac{dx}{x} + dx \log.\, x + \frac{d(1+\log.\, x)}{1+\log.\, x}$$

$$\frac{dv}{x^x(1+\log.\, x)} = dx\,(1+\log.\, x) + \frac{\frac{dx}{x}}{1+\log.\, x}$$

$$\frac{d^2u}{dx^2} = \frac{dv}{dx} + x^x(1+\log.\, x)^2 + \frac{1}{x} \cdot \frac{1+\log.\, x}{1+\log.\, x}$$

$$= x^x\,(1+\log.\, x)^2 + \frac{1}{x}.$$

Then to find the maximum or minimum values we must have

$$\frac{du}{dx} = x^x\,(1+\log.\, x) = 0,$$

which requires either $x^x = 0$, or $1 + \log.\, x = 0$. The first value makes $x = 0$, and also renders

$$\frac{d^2u}{dx^2} = 0 + \infty,$$

and it will be easily seen that all the successive differentials will involve the powers of $\frac{1}{x}$, and will consequently all become infinite when $x = 0$. Hence this value gives neither a maximum nor minimum.

The second gives $\log.\, x = -1$, and therefore

$$\log.\, \frac{1}{x} = 1.$$

Whence $\frac{1}{x} = \epsilon$, and $x = \frac{1}{\epsilon}$.

With this value $\frac{d^2u}{dx^2} = 0 + \epsilon$, which being positive gives

$$u = \left(\frac{1}{\epsilon}\right)^{\frac{1}{\epsilon}} = \text{a minimum}.$$

Or we thus find the number x, *which has the least* x*th power.*

(11.) In a somewhat similar manner we may solve the problem, to find *the number* x *whose* x*th root is the greatest possible.* In this case we have

$$u = x^{\frac{1}{x}};$$

or, taking the logarithms,

$$\log. u = \frac{1}{x}\log. x.$$

Whence by differentiating

$$\frac{du}{u} = -\frac{dx}{x^2}\log. x + \frac{1}{x}\frac{dx}{x}$$

$$= dx\frac{(1-\log. x)}{x^2}$$

$$\frac{du}{dx} = x^{\left(\frac{1}{x}-2\right)}(1-\log. x).$$

Again, $\frac{d^2u}{dx^2} = \left(\frac{1}{x}-2\right)x^{\left(\frac{1}{x}-3\right)}(1-\log. x) + x^{\left(\frac{1}{x}-2\right)}\left(-\frac{1}{x}\right)$

$$= x^{\left(\frac{1}{x}-3\right.}\left[\left(\frac{1}{x}-2\right)\left(1-\log. x\right)-1\right].$$

To make $\frac{du}{dx}=0$, either $x=0$, or $\log. x = 1$ and $\therefore x = \epsilon$.

The first value makes $\frac{d^2u}{dx^2}=0$.

The second gives it $= \epsilon^{\left(\frac{1}{\epsilon}-3\right)}[0-1]$, or makes the second differential negative; consequently $x=\epsilon$ gives a *maximum* value of the function.

(12.) Another function of the same kind is

$$u = \frac{x}{\log. x}.$$

Whence u log. $x = x$, and $\frac{u}{x} = \frac{1}{\log. x}$

$$du \log. x + u\frac{dx}{x} = dx$$

$$\therefore \frac{du}{dx} = \frac{1 - \frac{u}{x}}{\log. x} = \frac{\log. x - 1}{(\log. x)^2}$$

$$\frac{d^2u}{d^2} = \frac{(\log. x)^2 \frac{1}{x} - (\log. x - 1)\, 2 \log. x \frac{1}{x}}{(\log. x)^4}$$

$$= \frac{2 - \log. x}{x\,(\log. x)^3}.$$

To have $\frac{du}{dx} = 0$, $\log. x = 1$, or $x = \epsilon$.

This value gives

$$\frac{d^2u}{dx^2} = \frac{2-1}{\epsilon},$$

a ***positive*** value; consequently $x = \epsilon$ belongs to a ***minimum***, or we have the solution of the problem, ***to find the number which bears the least ratio to its logarithm.***

These, and some other remarkable properties of the same kind, are demonstrated by Euler, Diff. Calc. II. 272, &c.

(13.) Let $u = \sin. x \cos. (a - x)$; then we have,

$$\frac{du}{dx} = \cos. x \cos. (a - x) + \sin. x. \sin. (a - x)$$

$$= \cos. [x - (a - x)] = \cos. [2x - a]$$

$$\frac{d^2u}{dx^2} = -2 \sin. (2x - a).$$

The first differential $= 0$, if

$$x = \frac{a}{2} + \frac{\pi}{4}, \text{ or } \frac{a}{2} - \frac{\pi}{4}$$

This first value gives the original expression

$$u = \sin.\left(\frac{a}{2} + \frac{\pi}{4}\right) \cos.\left[a - \left(\frac{a}{2} + \frac{\pi}{4}\right)\right]$$

$$= \sin.\tfrac{1}{2}\left(a + \frac{\pi}{2}\right) \cos.\tfrac{1}{2}\left(a - \frac{\pi}{2}\right)$$

$$= \tfrac{1}{2}\left(\sin. a + \sin.\frac{\pi}{2}\right) = \frac{1 + \sin. a.}{2}.$$

Also the second differential has its sign dependant on that of $(2x - a)$: which in this instance

$$= a + \frac{\pi}{2} - a = +\frac{\pi}{2},$$

the sign of the whole is therefore negative: or this value belongs to a maximum.

If we take the second value of x, we shall find by a similar process the original expression

$$u = \frac{1 - \sin. a}{2};$$

and the second differential in this case becomes

$$-2 \sin.\left(-\frac{\pi}{2}\right),$$

or is positive: it therefore belongs to a minimum.

MAXIMA AND MINIMA OF TWO VARIABLES.

(1.) To divide a quantity a into three parts; x, y, and $a-x-y$, such that the product $x^m y^n (a-x-y)^p$ shall be a maximum. We have therefore

$$u = x^m y^n (a-x-y)^p.$$

And differentiating in respect of x,

$$\frac{du}{dx} = mx^{m-1} y^n (a-x-y)^p - x^m y^n p (a-x-y)^{p-1}$$

$$= [m x^{m-1} y^n (a-x-y) - x^m y^n p] (a-x-y)^{p-1};$$

and making this $=0$, we have the factor

$$[x^{m-1} y^n (ma-mx-my-px] = 0.$$

In like manner differentiating in respect of y we deduce by the same process,

$$\frac{du}{dy} = \left(ny^{n-1} x^m (a-x-y) - x^m y^n p\right) (a-x-y)^{p-1}.$$

Whence $y^{n-1} x^m [na-nx-ny-px] = 0$.

Then since in these two expressions the factors involving the constants are each $=0$, their sum is also $=0$; or we have

$$ma-mx-my-px+na-nx-ny-py = 0,$$

and

$$ma+na = mx+px+nx+my+ny+py.$$

Hence

$$ma + na = (m + p + n)x + (m + n + p)y;$$

or, $$ma - (m + p + n)\,x = na - (m + n + p)\,y,$$

and each of these being $= 0$, we deduce

$$x = \frac{ma}{m + n + p}, \qquad y = \frac{na}{m + n + p},$$

and therefore

$$a - x - y = \frac{a(m + n + p) - ma - na}{m + n + p}$$

$$= \frac{ap}{m + n + p};$$

or for brevity writing the denominator $= q$, we have

$$x = \frac{ma}{q} \qquad y = \frac{na}{q} \qquad a - x - y = \frac{pa}{q}.$$

And to discover whether these values belong to a maximum or a minimum, we must substitute them in the general expressions of

$$\frac{d^2u}{dx}, \quad \frac{d^2u}{dx\,dy}, \quad \frac{d^2u}{dy}.$$

To obtain the differential coefficients of the second order we take the value of the first:

$$\frac{du}{dx} = \left(x^{m-1}\,(a - x - y)^{p-1}\right)\left(ma - nx - my - px\right)y^n.$$

Whence we obtain, in respect of x,

$$\frac{d^2u}{dx^2} = \left\{\begin{array}{l} \left((m-1)\,x^{m-2}\,(a-x-y)^{p-1} - x^{m-1}(p-1)(a-x-y)^{p-2}\right) \\ (y^n)\,(ma - mx - my - px) \\ + x^{m-1}\,(a-x-y)^{p-1}\,y^n\,(-m-p). \end{array}\right.$$

Hence substituting the values of x, y, and $a - x - y$, and observing that the factor

$$ma - mx - my - px = \left(m\frac{pa}{q} - p\frac{ma}{q}\right) = 0.$$

The expression becomes,

$$\frac{d^2u}{dx^2} = -(m+p)\left(\frac{ma}{q}\right)^{m-1}\left(\frac{na}{q}\right)^{n}\left(\frac{pa}{q}\right)^{p-1}.$$

And in like manner differentiating for y,

$$\frac{d^2u}{dy^2} = -(n+p)\left(\frac{ma}{q}\right)^{m}\left(\frac{na}{q}\right)^{n-1}\left(\frac{pa}{q}\right)^{p-1}.$$

And again differentiating $\frac{du}{dx}$ in respect of y, we have

$$\frac{d^2u}{dx\,dy} = \left\{\begin{array}{l} [-x^{m-1}(p-1)(a-x-y)^{p-2}y^n + x^{m-1}(a-x-y)^{p-1}n\,y^{n-1}] \\ \qquad\qquad\qquad [ma-mx-my-px] \\ \quad + x^{m-1}y^n(a-x-y)^{p-1}(-m); \end{array}\right.$$

which, substituting the values of x, y, and $a-x-y$ becomes

$$\frac{d^2u}{dx\,dy} = -m\left(\frac{ma}{q}\right)^{m-1}\left(\frac{na}{q}\right)^{n}\left(\frac{pa}{q}\right)^{p-1}.$$

Of these expressions the first and second are those designated by A and C, and the third by B, in the general investigation of maxima and minima of two variables, [Application of Calc. to Curves, p. 268]; and it is evident that A and C have both the same sign, and are negative: and we have also the condition that $AC > B^2$, since

$$AC = (m+p)(n+p)\cdot\left(\frac{ma}{q}\right)^{2m-1}\left(\frac{na}{q}\right)^{2n-1}\left(\frac{pa}{q}\right)^{2p-2},$$

and

$$B^2 = m^2\left(\frac{ma}{q}\right)^{2m-2}\left(\frac{na}{q}\right)^{2n}\left(\frac{pa}{q}\right)^{2p-2};$$

which is easily put into a form in which it may be compared with AC by writing it

$$=m^2 \cdot \frac{q}{ma} \cdot \frac{na}{q} \cdot \left(\frac{ma}{q}\right)^{2m-1} \left(\frac{na}{q}\right)^{2n-1} \left(\frac{pa}{q}\right)^{2p-2}.$$

Thus we have only to compare the factors

$$(m+p)(n+p) = mn + pq;$$

and $$\frac{m^2 q\, n\, a}{m\, a\, q} = mn;$$

or we have evidently AC greater than B^2. These conditions then being fulfilled by the values above found for x, y, and $a-x-y$, it follows that they correspond to a *maximum*.

(2.) To find the maximum and minimum values of the function

$$u = y^4 - 8y^3 + 18y^2 - 8y + x^3 - 3x^2 - 3x.$$

Here, differentiating in respect of y, we have

$$\frac{du}{dy} = 4y^3 - 3 \cdot 8 \cdot y^2 + 2 \cdot 18\, y - 8:$$

or we may take

$$y^3 - 6y^2 + 9y - 2 = 0,$$

which is easily resolved into its factors by observing that it may be written

$$y^3 - 6y^2 + 12y - 8 - 3y + 6 = 0,$$

which $= (y-2)^3 - 3(y-2) = (y-2)\left((y-2)^2 - 3\right)$;

or the factors are

$$(y-2)$$

and $$(y-2)^2 - 3 = [(y-2) + \sqrt{3}]\,[(y-2) - \sqrt{3}].$$

Again, differentiating in respect of x, we find

$$\frac{du}{dx} = 3x^2 - 2 \cdot 3 \cdot x - 3,$$

or, $$x^2 - 2x - 1 = 0.$$

This is evidently

$$=(x-1)^2-2$$
$$=(x-1+\sqrt{2})\,(x-1-\sqrt{2}).$$

Thus the values of y for maxima or minima are

$$y=2, \qquad y=2+\sqrt{3}, \qquad y=2-\sqrt{3},$$

and those of x,

$$x=1+\sqrt{2}, \qquad x=1-\sqrt{2}.$$

Proceeding to a second differentiation in respect of y, we have

$$\frac{d^2u}{dy^2}=12y^2-48y+36 \qquad (A),$$

and, in respect of x,

$$\frac{d^2u}{dx^2}=6x-6 \quad . \quad . \quad . \quad (C.)$$

But we cannot differentiate $\frac{du}{dy}$ in respect of x, since it in this case involves no function of x. Then we must write

$$\frac{d^2u}{dy\,dx}=0. \quad . \quad . \quad . \quad . \quad . \quad (B.)$$

If we now take $y=2$ and substitute in (A), we shall readily find

$$A=-12.$$

In like manner substituting $x=1-\sqrt{2}$ in (C), we find

$$C=-6\sqrt{2}.$$

Hence A and C being both negative, and AC obviously greater than B^2, these values correspond to a maximum: or substituting them in the expression for u, we shall find on expanding the powers and multiplying the terms,

$$u=3+4\sqrt{2}$$

for a maximum value.

Again, if we take the value $y=2-\sqrt{3}$, or $=2+\sqrt{3}$, in either case it will be found that the value of (A) becomes by substitution

$$(A)=24.$$

If we take also $x=1+\sqrt{2}$, we shall find in like manner

$$(C)=6\sqrt{2}.$$

These results being both positive, and $AC>B^2$ as before, they belong to a minimum; or substituting in u,

$$u=-6-4\sqrt{2}$$

is a minimum value of the function.

If we take $y=2\pm\sqrt{3}$ and $x=1-\sqrt{2}$, we shall have as before

$$A=24, \text{ and } C=-6\sqrt{2},$$

where the signs being different, these values give neither a maximum nor a minimum.

VANISHING FRACTIONS.

We will here subjoin a few examples illustrative of the theory of vanishing fractions, in addition to those given in the former part of this work.

(1.) The value of the fraction

$$\frac{(x^2-a^2)^{\frac{3}{2}}}{(x-a)^{\frac{3}{2}}}.$$

when $x=a$, cannot be found by the differentiation of the numerator and denominator, it being obvious that by that process we should only obtain a continued series of powers of the binomials, which would all vanish on the supposition $x=a$; we must therefore proceed by substituting $x=a+h$, and developing. Whence we have

$$\frac{(a^2+2ah+h^2-a^2)^{\frac{3}{2}}}{(a+h-a)^{\frac{3}{2}}}=\frac{(2a+h)^{\frac{3}{2}}h^{\frac{3}{2}}}{h^{\frac{3}{2}}};$$

which when $h=0$, or when $x=a$, becomes

$$=(2a)^{\frac{3}{2}}.$$

(2.) In like manner, if we wish to find the value of the fraction

$$\frac{x^{\frac{1}{2}}-a^{\frac{1}{2}}+(x-a)^{\frac{1}{2}}}{(x^2-a^2)^{\frac{1}{2}}},$$

when $x=a$, we must proceed by substituting $(a+h)$ in the place of (x), when the expression will appear under the form

$$\frac{\sqrt{a+h} - \sqrt{a} + \sqrt{h}}{\sqrt{2ah+h^2}}$$

which, by expanding the square root of the binomial, becomes

$$\frac{\sqrt{a} + \frac{1}{2}a^{-\frac{1}{2}}h + \&c. - \sqrt{a} + \sqrt{h}}{\sqrt{h}\,\sqrt{2a+h}}.$$

Or (dividing by $\sqrt{h}$) it is reduced to

$$\frac{\frac{1}{2}a^{-\frac{1}{2}}\sqrt{h} + \&c. + 1}{\sqrt{2a+h}};$$

which when $x=a$, or when $h=0$, becomes

$$=\frac{1}{\sqrt{2a}}.$$

(3.) The transcendental function,

$\dfrac{x^x - x}{1-x+\log. x}$, when $x=1$, becomes $\dfrac{1-1}{1-1+0}=\dfrac{0}{0}$.

To find its value, let $x=1+h$;

then the function becomes

$$u=\frac{(1+h)^{(1+h)}-(1+h)}{1-(1+h)+\log.(1+h)}=\frac{(1+h)^{(1+h)}-(1+h)}{\log.(1+h)-h}.$$

Developing the first term of the numerator by the binomial theorem, it becomes

$$1+(1+h\;\;+\frac{(1+h)(1+h-1)}{1\,.\,2}h^2+\&c.,$$

or the whole numerator will be

$$1+(1+h)h+\frac{(1+h)\ h}{1\ .\ 2}\,h^2+\ \ldots\ldots\ -(1+h)$$
$$=1+(1+h)\ (h-1)\ +\ \&c.$$
$$=1+h^2-1\qquad\qquad +\ \&c.$$
$$=h^2\ +\ \frac{(1+h)}{1\ .\ 2}\,h^3+\&c.$$

And for the denominator taking log. $(1+h)$ by the formula, (Diff. Cal. form 26.)

$$\log.\ (1+h)=h-\tfrac{1}{2}\,h^2+\tfrac{1}{3}\,h^3-\ \&c.$$

and the whole denominator is $-\ \frac{1}{2}\,h^2+\frac{1}{3}\,h^3-\&c.$

hence the whole fraction

$$u=\frac{h^2+\dfrac{1+h}{1\ .\ 2}\,h^3+\&c.}{-\frac{1}{2}\,h^2+\frac{1}{3}\,h^3-\&c.}$$

$$=\frac{1+\dfrac{1+h}{1\ .\ 2}\ .\ h+}{-\frac{1}{2}+\frac{1}{3}h-}$$

which when $h=0$, becomes

$$(u)=\frac{1+0}{-\frac{1}{2}+0}=-2.$$

ON THE CONNECTION BETWEEN LOGARITHMIC AND CIRCULAR FUNCTIONS.

(1.) Between these two classes of transcendental functions a remarkable and intimate relation subsists, which we shall proceed to explain as follows:

Let $y = \sin. x$, whence $\sqrt{1-y^2} = \cos. x$.

Then by differentiation we have,

$$dy = \sqrt{1-y^2}\, dx,$$

and $d\sqrt{1-y^2} = -ydx$.

Multiplying the first of these equations by $\sqrt{-1}$, it becomes,

$$\sqrt{-1} \, . \, dy = \sqrt{1-y^2} \, . \, \sqrt{-1} \, . \, dx;$$

and adding this and the last equation, they give

$$d\sqrt{1-y^2} + \sqrt{-1} \, . \, dy = (\sqrt{1-y^2} \, . \, \sqrt{-1} - y)\, dx;$$

which may be written,

$$d\,(\sqrt{1-y^2} + \sqrt{-1} \, . \, y) = (\sqrt{1-y^2} + \sqrt{-1} \, . \, y)\sqrt{-1}\, dx,$$

whence we have,

$$\frac{d\,(\sqrt{1-y^2} + \sqrt{-1} \, . \, y)}{\sqrt{1-y^2} + \sqrt{-1} \, . \, y} = \sqrt{-1}dx.$$

But the first member of this equation is evidently the differential of the Napierian logarithm of the quantity

$$(\sqrt{1-y^2} + \sqrt{-1} \, . \, y);$$

or we have,

$$\log.\ (\sqrt{1-y^2}+\sqrt{-1}\,.\,y)=\sqrt{-1}\,.\,x\,;$$

whence, expressing the quantities by their trigonometrical designations,

$$\cos.\ x+\sqrt{-1}\,.\,\sin.\ x=\epsilon^{x\sqrt{-1}}\quad\ldots\ldots\ldots\quad(1)$$

If we had taken x with the negative sign, the same process would have given us

$$\cos.\ x-\sqrt{-1}\ \sin.\ x=\epsilon^{-x\sqrt{-1}}$$

Adding this equation to the last, we have,

$$\cos.\ x=\frac{\epsilon^{x\sqrt{-1}}+\epsilon^{-x\sqrt{-1}}}{2}\quad\ldots\ldots\ldots\ldots\quad(2)$$

And subtracting,

$$\sqrt{-1}\ \sin.\ x=\frac{\epsilon^{x\sqrt{-1}}-\epsilon^{-x\sqrt{-1}}}{2}$$

$$\text{or, }\sin.\ x=\frac{\epsilon^{x\sqrt{-1}}-\epsilon^{-x\sqrt{-1}}}{2\sqrt{-1}}\quad\ldots\ldots\ldots\ldots\quad(3)$$

(2.) These formulæ might also be deduced in another way, by assuming the developements of sin. x and cos. x, and comparing them with those of $\epsilon^{x\sqrt{-1}}$ and $\epsilon^{-x\sqrt{-1}}$. This was in fact the method followed by Euler, who originally investigated these remarkable formulæ: [Mém. de l'Acad. de Berlin, vol. VII.] the method given above is adopted by Lagrange, [Calcul des Fonctions, leçon X.] who characterizes these theorems as among the most beautiful discoveries of the age. And so in fact they may be justly regarded, whether we consider the elegance of their form, the singular nature of the relation exhibited, or

their numerous and important applications; some of which we shall now proceed to consider.

(3.) One direct and important deduction from these formulæ is as follows: since x may represent any arc, let it be replaced by nx, then we have in form (1),

$$\epsilon^{\pm nx\sqrt{-1}} = \cos. nx \pm \sqrt{-1} \sin. nx.$$

But it is also evident that

$$\epsilon^{\pm nx\sqrt{-1}} = (\epsilon^{\pm x\sqrt{-1}})^n;$$

which is therefore $= (\cos. x \pm \sqrt{-1}, \sin. x)^n$; whence we deduce,

$$(\cos. x \pm \sqrt{-1} \sin. x)^n = \cos. nx \pm \sqrt{-1} \sin. nx \quad . \quad . \quad (4)$$

This formula was discovered by De Moivre, and has very extensive applications: in fact, in this and the preceding formulæ almost the whole science of analytical trigonometry may be said to be included. For a complete view of this subject, the student is referred to Hind's Trigonometry, ch. vi.; or to Lardner's, part iii.

(4.) We will here give the application of De Moivre's formula for expressing the sine and cosine of multiple arcs in terms of the powers of the sine and cosine of the simple arc; which is effected as follows:

If we take the formulæ,

$$(\cos. x + \sqrt{-1}, \sin. x)^n = \cos. nx + \sqrt{-1}, \sin. nx,$$

$$(\cos. x - \sqrt{-1}, \sin. x)^n = \cos. nx - \sqrt{-1}, \sin. nx,$$

and expand the first side of each equation, we have,

$$\cos.^n x + n \cos.^{n-1} x\sqrt{-1}, \sin. x$$
$$+ \frac{n \,.\, n-1}{2}, \cos.^{n-2} x\,(-1) \sin.^2 x + \&c.$$

$$\cos.^n x - n \cos.^{n-1} x\sqrt{-1}, \sin. x$$
$$+ \frac{n \,.\, n-1}{2}, \cos.^{n-2} x\,(-1), \sin.^2 x - \&c.$$

Adding the two formulæ, we find,

$$2\cos.nx = (\cos.x + \sqrt{-1},\sin.x)^n + (\cos.x - \sqrt{-1},\sin.x)^n$$

and substituting the expansions, adding them, and dividing by two, we obtain,

$$\cos.nx = \cos.^n x - \frac{n.n-1}{2}\cos.^{n-2}x\sin.^2 x + \&c. \quad . \; . \; . \quad (5)$$

And subtracting,

$$2\sqrt{-1}\sin.nx = 2\sqrt{-1}\,n\cos.^{n-1}x\sin.x$$
$$+2.\frac{n(n-1)(n-2)}{2.3}\cos.^{n-3}(-1)^{\frac{3}{2}}\sin.^3 x + \&c.$$

whence,

$$\sin.nx = n\cos.^{n-1}x\sin.x$$
$$-\frac{n.(n-1)(n-2)}{2.3}\cos.^{n-3}\sin.^3 x + \&c. \quad . \; . \; . \; . \; . \; . \quad (6)$$

(5.) These formulæ also enable us to find the developement of $\cos.^m x$ in terms of the multiple arcs, without employing the powers of the sine and cosine. For this purpose, let us assume

$$\cos.x + \sin.x\sqrt{-1} = u \quad . \; . \; . \; . \; . \; . \; . \; . \; . \; . \quad (7)$$

$$\cos.x - \sin.x\sqrt{-1} = v \quad . \; . \; . \; . \; . \; . \; . \; . \; . \; . \quad (8)$$

These equations being added, give

$$\cos.x = \frac{1}{2}(u+v);$$

and consequently

$$\cos.^m x = \frac{1}{2^m}(u+v)^m, \cos.^m x = \frac{1}{2^m}(v+u)^m;$$

developing these binomials, we obtain

$$\cos.^m x = \frac{1}{2^m}\left(u^m + mu^{m-1}v + m\ .\ \frac{m-1}{2}u^{m-2}v^2 + \&c.\right)$$

$$\cos.^m x = \frac{1}{2^m}\left(v^m + mv^{m-1}u + m\ .\ \frac{m-1}{2}v^{m-2}u^2 + \&c.\right);$$

and adding these equations, we find

$$2^{m+1}\cos.^m x = u^m + v^m + muv\left(u^{m-2} + v^{m-2}\right)$$
$$+ m\ .\ \frac{m-1}{2}u^2v^2\left(u^{m-4} + v^{m-4}\right) + \&c. \quad (9)$$

But from the formulæ (7) and (8) we deduce

$$u^m = (\cos. x + \sqrt{-1}\sin. x)^m,\ v^m = (\cos. x - \sin. x\sqrt{-1})^m,$$

and substituting on the second sides of these equations their values given by the formulæ (2) and (6), we have

$$\left.\begin{array}{l} u^m = \cos. mx + \sin. mx\sqrt{-1}, \\ v^m = \cos. mx - \sin. mx\sqrt{-1}, \end{array}\right\} \ldots\ldots \quad (10);$$

whence adding these expressions,

$$u^m + v^m = 2\cos. mx, \text{ and } u^m v^m = 1,$$

and consequently

$$\begin{array}{l} \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots .\ uv = 1, \\ u^{m-2} + v^{m-2} = 2\cos.(m-2)x,\ u^{m-2}v^{m-2} = 1, \\ u^{m-4} + v^{m-4} = 2\cos.(m-4)x,\ u^{m-4}v^{m-4} = 1, \\ \quad \&c. \quad = \quad \&c. \qquad\qquad\qquad \&c. \end{array}$$

Substituting these values in the equation (9), we shall find

$$\cos.^m x = \frac{1}{2^{m+1}}\left[2\cos. mx + 2m\cos.(m-2)x\right.$$
$$\left. + 2m\ .\frac{(m-1)}{1\ .\ 2}\cos.(m-4)x + \&c.\right] \ldots\ldots \quad (11).$$

By a similar process we may find the developement of $\sin.{}^m x$: for this purpose, subtracting the equation (8) from the equation (7), we obtain,

$$2 \sin. x\sqrt{-1} = u - v,$$

$$\text{and therefore } \sin. x = \frac{u-v}{2\sqrt{-1}};$$

and raising the two sides of this equation to the power m, we shall have

$$\sin.{}^m x = \frac{1}{(2\sqrt{-1})^m}(u-v)^m.$$

Now if m be equal to an even number $2p$, we have

$$(u-v)^{2p} = [(u-v)^2]^p = [(v-u)^2]^p = (v-u)^{2p};$$

whence

$$(u-v)^m = (v-u)^m.$$

Then developing the equations

$$\sin.{}^m x = \frac{1}{(\sqrt{2-1})^m}(u-v)^m,$$

$$\text{and } \sin.{}^m x = \frac{1}{(2\sqrt{-1})^m}(v-u)^m,$$

and proceeding as we did above, we shall find

$$\sin.{}^{mx} = \frac{1}{(2\sqrt{-1})^m}(\cos. mx - m \cos. (m-2)x$$

$$+ m \,.\, \frac{m-1}{2} \cos. (m-4)\, x - \&c.];$$

the imaginary quantity $2\sqrt{-1}$ being raised to an even power will disappear.

If m be equal to an odd number $2p+1$, we shall have

$$(u-v)^{2p+1}=(u-v)^{2p}\times(u-v)=(v-u)^{2p}\times-(v-u)$$
$$=-(v-u)^{2p+1},$$

whence

$$(u-v)^m=-(v-u)^m,$$

and

$$\sin.^m x=\frac{(u-v)^m}{(2\sqrt{-1})^m},\ \sin.^m x=\frac{(v-u)^m}{(2\sqrt{-1})^m}\quad . \ . \ . \ (12)$$

developing $(u-v)^m$ and $(v-u)^m$ by the binomial theorem, and substituting these developements in the equations (12), added together, we shall have

$$2\sin.^m x=\frac{1}{(2\sqrt{-1})^m}[u^m-v^m-m\,.\,uv(u^{m-2}-v^{m-2})+\&c.](13)$$

Subtracting then the equations (10) from each other, multiplying the same equations together, and observing that the second operation gives us the sum of the squares of the sine and cosine of mx, which is equivalent to unity, we shall find

$$u^m-v^m=2\sin.\,mx\sqrt{-1},\ u^m v^m=1\,;$$

and proceeding therefore in the same manner as before, we shall change the equation (13) into

$$\sin.^m x=\frac{1}{2\,(2\sqrt{-1})^{m-2}}[\sin.\,mx-m\sin.\,(m-2)\,x$$
$$+\frac{m\,.\,(m-1)}{2}\sin.\,(m-4)\,x\ \&c.]$$

Since, on this hypothesis. m is odd, the power $m-1$, to which the quantity $2\sqrt{-1}$ is raised, will be even, and the imaginary quantity will consequently disappear.

(6.) Another application of these formulæ is to the discovery of the factors of binomials of the form $x^n\pm a^n$: an inquiry which becomes necessary in the

integration of rational fractions, first made by Cotes in his Harmonia Mensurarum, and subsequently extended by De Moivre.

If we make $x = ay$, the function $x^n \pm a^n$ is transformed into $a^n (y^n \pm 1)$, and for the resolution of this quantity into its factors we have to solve the equation

$$y^n \pm 1 = 0.$$

Now if we take an arc z, and suppose

$$y^n = (\cos. z \pm \sqrt{-1} \sin. z)^n,$$

this, by what has preceded, is readily seen to be

$$= \cos. nz \pm \sqrt{-1} \sin. nz.$$

And since, if m be a whole number, we have

$$\sin. m\pi = 0 \quad \cos. m\pi = \pm 1,$$

(the sign being + or − according as m is an even or an odd number :) then if nz become $= m\pi$, we shall have

$$y^n = \pm 1.$$

Hence we see that the value

$$y = \cos. z \pm \sqrt{-1} \sin. z \quad \ldots\ldots\ldots\ldots \quad (14)$$

satisfies the conditions of the equation, when for z we substitute its value derived from the equation $nz = m\pi$, or $z = \dfrac{m\pi}{n}$.

But it will be more convenient to adopt a notation which shall distinguish the odd from the even multiples of π. For this purpose we will write the even multiplier $= 2m$, and the odd $= 2m + 1$, or suppose nz successively equal to

$$2m\pi, \text{ or to } (2m + 1)\,\pi,$$

Thus the expressions become for *even* values, substituting for $z = \frac{2m\pi}{n}$,

$$y^n = +1, \quad y = \cos.\frac{2m\pi}{n} + \sqrt{-1}\,\sin.\frac{2m\pi}{n},$$

and for the *odd* values

$$y^n = -1, \quad y = \cos.\frac{(2m+1)\pi}{n} + \sqrt{-1}\,\sin.\frac{(2m+1)\pi}{n} \quad \ldots \quad (15).$$

By substituting successive whole numbers for m, we shall have a series of values for y in each case, which give the factors of the first degree of the quantity $(y^n - 1)$, which are all comprised under the two general imaginary expressions,

$$\left(y - \cos.\frac{2m\pi}{n}\right) - \sqrt{-1}\,\sin.\frac{2m\pi}{n}$$

$$\left(y - \cos.\frac{2m\pi}{n}\right) + \sqrt{-1}\,\sin.\frac{2m\pi}{n}.$$

And the product of these, which is easily found to be

$$y^2 - 2y\cos.\frac{2\,m\pi}{n} + 1 \quad \ldots\ldots \quad (16)$$

comprehends all the real factors of the second degree.

In the same manner for the quantity y^{n+1} the real factors of the second degree are comprised in the expression

$$y^2 - 2y\cos.\frac{(2m+1)\pi}{n} - 1 \quad \ldots\ldots \quad (17)$$

For example; let us take the quantity $(y^6 - 1)$, or $n = 6$. The factors of the first degree will be obtained as follows by making successively

$m=0$, which gives $\left\{\begin{array}{l} y-(\cos. 0 \pm \sqrt{-1} \sin. 0) \\ =y-1 \end{array}\right.$

$m=1$ . . . . $y-\left(\cos. \frac{2\pi}{6} \pm \sqrt{-1} \sin. \frac{2\pi}{6}\right)$

$m=2$ . . . . $y-\left(\cos. \frac{4\pi}{6} \pm \sqrt{-1} \sin. \frac{4\pi}{6}\right)$

$m=3$ . . . . $\left\{\begin{array}{l} y-\left(\cos. \frac{6\pi}{6} \pm \sqrt{-1} \sin. \frac{6\pi}{6}\right) \\ \text{or } y-(\cos. \pi \pm \sqrt{-1} \sin. \pi) \\ =y-(-1)=y+1. \end{array}\right.$

With this value of m the series terminates, since by going on we should only have the same values recurring. And in a similar way we have the factors of the second degree, arising from the products of the *imaginary* factors of the first degree.

$$y^2-2y \cos. \frac{2\pi}{6}+1$$

$$y^2-2y \cos. \frac{4\pi}{6}+1.$$

whilst the factor of the second degree, which is the product of the *real* factors $(y-1)$ and $(y+1)$, is obviously

$$y^2-1.$$

In a similar manner for the case y^6+1, we have the simple factors

$$y-\left(\cos. \frac{\pi}{6} \pm \sqrt{-1} \sin. \frac{\pi}{6}\right)$$

$$y-\left(\cos. \frac{3\pi}{6} \pm \sqrt{-1} \sin. \frac{3\pi}{6}\right), \text{ or } y \mp \sqrt{-1}$$

$$y-\left(\cos. \frac{5\pi}{6} \pm \sqrt{-1} \sin. \frac{5\pi}{6}\right).$$

And the factors of the second degree

$$y^2 - 2y \cos. \frac{\pi}{6} + 1$$

$$y^2 + 1$$

$$y^2 - 2y \cos. \frac{5\pi}{6} + 1.$$

(7.) The same mode of solution is also applicable to functions of the form

$$x^{2n} - 2px^n + q.$$

For this expression being solved as a quadratic, we shall have the two factors included in the form

$$x^n - (p \pm \sqrt{p^2 - q}),$$

which will be real, if $p^2 > q$: and in that case, if we make

$$\pm a^n = p \pm \sqrt{p^2 - q},$$

we shall have these factors in the form

$$x^n \pm a^n,$$

which are then again resolvable into their factors, as we have already seen.

If we have $p^2 < q$, by writing

$$p = \alpha^n \qquad q = \beta^n \qquad x = \beta y,$$

the original form becomes

$$\beta^{2n} y^{2n} - 2\alpha^n \beta^n y^n + \beta^{2n}$$

$$= \beta^{2n} \left(y^{2n} - \frac{2\alpha^n}{\beta^n} y^n + 1\right).$$

But the condition $p^2 < q$, that is, $\alpha^{2n} < \beta^{2n}$, gives $\alpha^n < \beta^n$, or $\frac{\alpha^n}{\beta^n} < 1$. This fraction may therefore be represented by the cosine of a given arc δ, and the proposed function will be reduced to

$$\beta^{2n}(y^{2n}-2y^n\cos.\delta+1);$$

and we have only to resolve the equation

$$y^{2n}-2y^n\cos.\delta+1=0.$$

Whence we find immediately

$$y^n=\cos.\delta\pm\sqrt{-1}\sin.\delta;$$

and comparing this with the formula

$$y^n=\cos.nz\pm\sqrt{-1}\sin.nz,$$

we have $\cos.nz=\cos.\delta.\ \sin.nz=\sin.\delta:$

which conditions are fulfilled, if we suppose

$$nz=2m\pi+\delta,$$

where m is any whole number: since

$$\cos.(2m\pi-\delta)=\cos.\delta.\quad \sin.(2m\pi+\delta)=\sin.\delta.$$

Hence we have

$$z=\frac{2m\pi+\delta}{n}$$

and

$$y=\cos.\left(\frac{2m\pi+\delta}{n}\right)\pm\sqrt{-1}\sin.\left(\frac{2m\pi+\delta}{n}\right).$$

And the factors of the first degree of the function

$$y^{2n}-2y^n\cos.\delta+1$$

will consequently be comprehended in the formula

$$y-\left(\cos.\left(\frac{2m\pi+\delta}{n}\right)\pm\sqrt{-1}\sin.\left(\frac{2m\pi+\delta}{n}\right)\right)\quad .\ .\ (18).$$

If we had assumed the original expression with a different sign, or

$$x^{2n}+2px^n+q=0,$$

we should still assume

$$\frac{\alpha^n}{\beta^n}=\cos.\delta;$$

but we must take

$$y^{2n} - 2y^n \cos. (\pi - \delta) + 1,$$

since $\cos. (\pi - \delta) = -\cos. \delta$.

Hence there will result

$$\cos. nz = \cos. (\pi - \delta) \qquad \sin. nz = \sin. (\pi - \delta),$$

and consequently

$$nz = 2m\pi + \pi - \delta = (2m + 1)\ \pi - \delta.$$

For a full account of this subject the reader is referred to Lacroix's large treatise.

ON THE DEVELOPEMENT OF FUNCTIONS.

(1.) The differentiation of exponential functions being supposed established, and assuming that they may be developed in a series of powers of x, we may obtain their developement without Maclaurin's formula.

We will give only the process for thus obtaining ϵ^x, which will sufficiently exemplify the method: it is precisely similar to that used in deducing Taylor's theorem.

Let $\epsilon^x = 1 + ax + bx^2 + cx^3 + \&c.$

$$\text{then } \frac{d\epsilon^x}{dx} = \epsilon^x = a + 2bx + 3cx^2 + \&c.$$

and equating coefficients we have

$$a = 1, \quad b = \frac{a}{2} = \frac{1}{1.2}, \quad c = \frac{b}{3} = \frac{1}{1.2.3}, \ \&c.$$

$$\text{or, } \epsilon^x = 1 + x + \frac{x^2}{1.2} + \frac{x^3}{1.2.3} + \&c.$$

(2.) The following is another instance in which the process of differentiation is applied directly to the developement of a complex function.

Let $v = a + bx + cx^2 + dx^3 + \&c.$ . . . (1).
Where the coefficients are known:
and $u = \epsilon^v$,

which it is required to develope in a series of powers of x, with coefficients which are functions of a, b, c, &c.

First, since $dv = \dfrac{du}{u}$

$$\text{we have } \frac{du}{dx} = u\,\frac{dv}{dx}.$$

Then, in order to develope u, let us assume the following undefined series with indeterminate coefficients,

$$u = A + Bx + Cx^2 + Dx^3 + \&c. \quad . \quad . \quad . \quad (3).$$

We may proceed to determine these coefficients by the following process;

Differentiating equation (3) we have

$$\frac{du}{dx} = 0 + B + 2Cx + 3Dx^2 + \&c.$$

In like manner from equation (1)

$$\frac{dv}{dx} = 0 + b + 2cx + 3dx^2 + \&c.$$

Hence we have only in the form (2) to substitute these values, and the second side of that equation becomes

$$u\,\frac{dv}{dx} = [A + Bx + Cx^2 + \&c.]\ [b + 2cx + 3dx^2 + \&c.]$$

$$= \left\{ \begin{array}{l} Ab + Bbx + Cbx^2 - \&c. \\ + 2Acx + 2Bcx^2 + 2Ccx^3 + \&c. \\ + 3Adx^2 + 3Bdx^3 + 3Cdx^4 + \&c. \end{array} \right.$$

This series is therefore equal to

$$\frac{du}{dx} = B + 2Cx + 3Dx^2 + \&c.$$

Hence equating coefficients, we find

$$Ab = B$$
$$Bb + 2Ac = 2C$$
$$Cb + 2Bc + 3Ad = 3D$$
$$\&c. \qquad = \&c.$$

By making $x = 0$ the value of v is reduced to $v = a$, and consequently,

$$\epsilon^a = A.$$

Hence we have the coefficients

$$B = \epsilon^a b$$
$$C = \frac{\epsilon^a b^2 + 2\epsilon^a c}{2}$$
$$D = \frac{\frac{(\epsilon^a b + 2\epsilon^a c)b}{2} + 2\epsilon^a bc + 3\epsilon^a d}{3}$$
$$\&c. = \qquad \&c.$$

or we obtain the developement,

$$u = \epsilon^a + \epsilon^a bx + \frac{\epsilon^a b^2 - 2\epsilon^a c}{1 \, . \, 2} x^2$$
$$+ \frac{\epsilon^a b^2 + 6\epsilon^a bc + 6\epsilon^a d}{1 \, . \, 2 \, . \, 3} + \&c.$$

$$u = \epsilon^a \left(1 + bx + \frac{b^2 + 2c}{1 \, . \, 2} x^2 + \frac{b^2 + 6bc + 6d}{1 \, . \, 2 \, . \, 3} + \&c.\right)$$

In some complex functions this method is employed with advantage. Lagrange has applied it to obtain series for sin. nx and cos. nx in powers of sin. x and cos. x. [Calcul des Fonctions Leçons 10, 11.] Other series are investigated by Euler in this manner. [Diff. Calc. part II. cap. viii. art. 207.]

(3.) Some of the principal developements may be considered in the following point of view:

If we assume the following series,

$$u = 1 + \frac{x}{1} + \frac{x^2}{1.2} + \frac{x^3}{1.2.3} + \&c.$$

$$v = 1 - \frac{x^2}{1.2} + \frac{x^4}{1.2.3.4} - \&c.$$

$$w = \frac{x}{1} - \frac{x^3}{1.2.3} + \frac{x^5}{1.2.3.4.5} - \&c.$$

$$y = \frac{x}{1} + \frac{x^2}{2} + \frac{x^3}{3} + \frac{x^4}{4} + \&c.$$

$$z = 1 + \frac{\mu}{1} x + \frac{\mu}{1} . \frac{\mu - 1}{2} x^2 + \&c.$$

we may make the following inferences by the mere application of the principle of differentiation:

(1st.) Differentiating the series for u, we have

$$du = dx + \frac{2x dx}{1.2} + \frac{3x^2 dx}{1.2.3} + \&c.$$

$$\frac{du}{dx} = 1 + \frac{x}{1} + \frac{x^2}{1.2} + \&c. = u.$$

When $dx = \frac{du}{u}$, and therefore $x = \log. u$, or $u = \epsilon^x$.

(2dly.) In like manner from the series for v and w, we obtain

$$\frac{dv}{dx} = -w, \text{ and } \frac{dw}{dx} = v \quad \therefore -\frac{dv}{w} = \frac{dw}{v}.$$

Whence $2vdv + 2wdw = 0$, or these being respectively the differentials of v^2 and w^2, we have

$$v^2 + w^2 = 1.$$

And thence substituting in the value of dx,

$$dx = \frac{dw}{\sqrt{1 - w^2}},$$

which gives

$$x = \sin.^{-1} \omega \text{ and } x = \cos.^{-1} v.$$

3dly. From the series for y we derive,

$$\frac{dy}{dx} = 1 + \frac{2x}{2} + \frac{3x^2}{3} + \frac{4x^3}{4} + \&c.$$

which is evidently the developement of $\frac{1}{1-x}$.

Hence $dy = \frac{dx}{1-x} = \frac{-d(1-x)}{1-x}$;

and therefore $y = -\log.(1-x) = \log.\left(\frac{1}{1-x}.\right)$

4thly. From the series for z we obtain

$$\frac{dz}{dx} = \mu + \mu . \frac{\mu - 1}{2} 2x + \mu . \frac{(\mu-1)(\mu-2)}{1.2.3.} 3.x^2 + \&c.$$

Or multiplying each side by $(1+x.)$

$$\frac{dz}{dx}(1+x) = (1+x).\mu.\left(1 + (\mu-1)x + \frac{(\mu-1)(\mu-2)}{2}x^2 + \&c.\right)$$

$$= \mu\left(1 + x + \mu x - x + \mu x^2 - x^2 + \frac{\mu(\mu-1)}{2}x^2 - (\mu-1)x^2 + \&c.\right)$$

$$= \mu\left(1 + \mu x + \frac{\mu . \mu - 1}{2}x^2 + \&c.\right)$$

$$= \mu(z)$$

Hence we have

$$\frac{dz}{z} = \frac{\mu dx}{1+x} \therefore \log. z = \mu \log.(1+x),$$

and therefore . . . $z = (1+x)^\mu$.

We may easily see that the three first of these series are always converging; and the two last, when x is less than unity.

In the first series, since the denominators receive an increasing factor at each term, it is evident, that if the denominator of any term be d, that of the next receiving a new factor p, will become dp; that of the next will be $dp\,(p+1)$; which will be greater than dp^2: and the next, or $dp\,(p+1)\,(p+2)$ will be greater than dp^3, and so on. The numerators, on the other hand, increase only by the constant factor x; thus, at some term, the denominators will begin to increase faster than the numerators, whatever be the value of x; or the series is always converging.

This is true, *a fortiori*, of the second and third series, in which the terms are alternately positive and negative.

It is also evident in the fourth and fifth, when x is a fraction, whatever be the nature of μ.[a]

ON TAYLOR'S THEOREM, &c.

(4.) Considerable discussion has arisen respecting the generality of the theorems of Taylor and Maclaurin. To enter into a detailed account of the various investigations which have been made by analysts on this subject would carry us beyond the proper limits of an elementary treatise. But perhaps the following general observations may tend to place the subject in a sufficiently clear light to satisfy the inquiries of the student.

[a] These elegant deductions were communicated by the author of "the Theorems of Taylor and Maclaurin in a finite form." Oxford, 1830.

In the investigations of these theorems, given in the former part of this work, and in most elementary treatises, the developements are deduced in a general form without any particular limitation as to the values they may assume in particular cases, or as to the number of terms to which they may extend. Such developements may consist either of a finite or an infinite series of terms; and in the latter case may be either divergent or convergent; and if convergent, either slowly or rapidly so: and thus for the application of such forms to obtain *numerical results*, it is of course understood that they may give such results either accurately, or approximately, or not at all. Again, if we have a finite series, it may still happen, that by assigning some particular value to the variable, the developement may fail: as for example, some of the differential coefficients may become infinite, and the formula then gives us no real developement.

To take one of the instances often adduced; if we wish to develope

$$y = \sqrt[3]{x-a},$$

we have by differentiating, the first coefficient,

$$\frac{dy}{dx} = \frac{1}{3 \,.\, (x-a)^{\frac{2}{3}}} \text{ and so on successively,}$$

and the developement will be

$$f(x+h) = \sqrt[3]{x-a} + \frac{h}{3\,(x-a)^{\frac{2}{3}}} + \&c.$$

This holds good in general: but if we suppose x to become $= a$, the first term of the developement becomes $= 0$, and the second infinite, and the process will therefore for this particular case give us no result.

When we attempt to apply the general theorems to

particular cases, it becomes evident, whether any such failure is involved. But it is objected that we assert universally the truth of the equations which constitute these theorems, without any limitation; and the ordinary proofs given of them are also unaccompanied by any such limitations, and both are therefore fallacious. The assumption that the function can be expressed in a series of ascending powers of x or of h, has been said to be unwarrantable, and the form of the resulting expression such as asserts universally an equality, when in many cases no such equality really subsists. Hence some writers have employed themselves in investigating proofs in which these defects shall be avoided, and in exhibiting the theorem in such a form as shall include an indication of the precise extent to which it applies in every case. Such an investigation has been given by Cauchy in his "Infinitesimal Calculus:" it will however be unnecessary here to insert it, since a demonstration, in substance the same, accompanied by several important remarks, is accessible to all our readers, having been published in an anonymous tract entitled, "the Theorems of Taylor and Maclaurin in a finite "form: Oxford, 1830." An investigation with which it is presumed the most scrupulous inquirer must be satisfied.

But perhaps the difficulties may admit of being removed, and the question placed in a similar point of view, by only carefully considering *how much is actually implied or assumed* in the usual general statements and proofs of these developements, when treated as referring to the variable in an *indeterminate* value, and as being continued to an *undefined* number of terms, which *may become limited or not*, by the nature of particular cases; and by regarding these theorems

as referring rather to the *form* of the developement, than to its numerical equality with a *value* of the function; this being a more general relation, which may include under it, in particular cases, that of the sum of a series. Viewing the matter in this light, we shall use these formulæ without being misled by our assumption to expect finite results in every case.

(5.) We will proceed to some deductions and applications which may illustrate the nature and meaning of these theorems for the developement of functions.

Taylor's formula, by transferring to the first side of the equation the first term of the developement of $f(x+h,)$ becomes the developement of the finite difference of the function in its two values, corresponding to the values x and $x+h$.

If we suppose the term dx to be represented by a finite quantity, and to be equal to h, the formula becomes

$$y_2 - y = \frac{dy}{1} + \frac{d^2y}{1.2} + \frac{d^3y}{1.2.3} + \&c.$$

whence we see in what manner the difference of the function corresponding to an increment $h = dx$, is composed of the differentials of various orders relative to the same increment.

In the applications of Taylor's theorem, it is often convenient to make the hypothesis $h = -x$, in which case it becomes

$$f(x-x) = f(0) = u - \frac{du}{dx}\frac{x}{1} + \frac{d^2u}{dx^2}\frac{x^2}{1.2} - \frac{d^3u}{dx^3}\frac{x^3}{1.2.3} + \&c.$$

(6.) We have before given, by the application of Taylor's Theorem, the developement of the sine and cosine in terms of the arc: we shall here take the in-

verse problem of expressing the *arc* in terms of its trigonometrical functions; or, in other words, treating the arc as a function of the sine or cosine. This will of course be founded on the differentiation of the arc with respect to the sine or cosine which is immediately obtained from the former process: for if

$$u = \sin.^{-1} x,$$

we have $$\cos. u = \sqrt{1-x^2},$$

and from $$dx = du \sqrt{1-x^2}$$

we get $$du = \frac{dx}{\sqrt{1-x^2}}. \quad \dots\dots\dots \quad (1)$$

And in the same way if $u = \cos.^{-1} x$,

from $$dx = -du \sqrt{1-x^2}$$

we get $$du = -\frac{dx}{\sqrt{1-x^2}}. \quad \dots\dots\dots \quad (2)$$

Again, for the tangent, if $u = \tan.^{-1} x$,

from $$dx = du \sec.^2 x = du(1+x^2)$$

we get $$du = \frac{dx}{(1+x^2)}. \quad \dots\dots\dots \quad (3)$$

From these fundamental expressions we can readily proceed to the several developements of the arc in terms of the sine, cosine, and tangent.

1st. Let $u = \sin.^{-1} x$:

Then we have

$$\frac{du}{dx} = (1-x^2)^{-\frac{1}{2}},$$

$$\frac{d^2u}{dx^2} = -\frac{1}{2}(1-x^2)^{-\frac{3}{2}}(-2x) = x(1-x^2)^{-\frac{3}{2}},$$

$$\frac{d^3u}{dx^3} = (1-x^2)^{-\frac{3}{2}} - \frac{3}{2}x\,(1-x^2)^{-\frac{5}{2}}(-2x)$$

$$= (1-x^2)^{-\frac{3}{2}} + 3x^2(1-x^2)^{-\frac{5}{2}}$$

$$\frac{d^4u}{dx^4} = -\frac{3}{2}(1-x^2)^{-\frac{5}{2}}(-2x) + 2\,.\,3\,.\,x(1-x^2)^{-\frac{5}{2}} +$$

$$3x^2\left(-\frac{5}{2}\right)(1-x^2)^{-\frac{7}{2}}(-2x)$$

$$= (3+2\,.\,3)\,x\,(1-x^2)^{-\frac{5}{2}} + 3\,.\,5\,x^3(1-x^2)^{-\frac{7}{2}}$$

$$\left.\begin{aligned}\frac{d^5u}{dx^5} &= 9(1-x^2)^{-\frac{5}{2}} + 9x\left(-\frac{5}{2}\right)(1-x^2)^{-\frac{7}{2}}(-2x)\\ &+ 2.3.5.x(1-x^2)^{-\frac{7}{2}} + 3.5.x^3\left(-\frac{7}{2}\right)(1-x^2)^{-\frac{9}{2}}(-2x)\end{aligned}\right\}$$

$$\left.\begin{aligned}= 3\,.\,3\,.\,(1-x^2)^{-\frac{5}{2}} + (2\,.\,3\,.\,5 + 9\,.\,5)\,x^2(1-x^2)^{-\frac{7}{2}}\\ + 3\,.\,5\,.\,7\,x^4(1-x^2)^{-\frac{9}{2}}\end{aligned}\right\}$$

Making $x=0$ these coefficients become respectively

$$1\ ,\ 0\ ,\ 1\ ,\ 0\ ,\ 3\,.\,3\,,\ \&c.$$

and we have the developement

$$u = x + \frac{x^3}{1.2.3} + \frac{3^2.x^5}{1.2.3.4.5} + \frac{3^2.5^2x^7}{1.2.3.4.5.6.7} + \&c.$$

2dly. Let $u = \cos.^{-1}x$.

We thence derive successively,

$$\frac{du}{dx} = \frac{-1}{\sqrt{1-x^2}}$$

$$\frac{d^2u}{dx^2} = -\frac{\frac{x}{\sqrt{1-x^2}}}{1-x^2} = \frac{-x}{(1-x^2)^{\frac{3}{2}}}$$

$$\frac{d^3u}{dx^3}=\frac{-(1-x^2)^{\frac{3}{2}}+x\,.\,\frac{3}{2}(1-x^2)^{\frac{1}{2}}(-2x)}{(1-x^2)^3}$$

$$=\frac{-(1-x^2)-3x^2}{(1-x^2)^{\frac{5}{2}}}$$

&c. = &c.

When $x=0$ these expressions become

$$u=\cos.^{-1}0=\frac{\pi}{2}$$

$$\frac{du}{dx}=-1 \qquad \frac{d^2u}{dx^2}=0 \qquad \frac{d^3u}{dx^3}=-1 \text{ \&c.}$$

Whence Stirling's theorem gives us

$$\cos.^{-1}x=\frac{\pi}{2}-\frac{x}{1}-\frac{x^3}{1.2.3}-\frac{x^5}{1.2.3.4.5}-\text{\&c.}$$

3dly. Again if we take

$u=\tan.^{-1}x$

we find the several differential coefficients,

$$\frac{du}{dx}=\frac{1}{1+x^2}=(1+x^2)^{-1}$$

$$\frac{d^2u}{dx^2}=-2x\,(1+x^2)^{-2}$$

$$\frac{d^3u}{dx^3}=-2\,(1+x^2)^{-2}+8x^2(1+x^2)^{-3}$$

$$\frac{d^4u}{dx^4}=4(1+x^2)^{-3}2x-24x^2(1+x^2)^{-4}.2x+16x(1+x^2)^{-3}$$

$$=24x\,(1+x^2)^{-3}-48x^3\,(1+x^2)^{-4}$$

$$\frac{d^5u}{dx^5}=\begin{cases}24(1+x^2)^{-3}\;+(-3)24.x\,(1+x^2)^{-4}2x\\ -(3.48x^2\,(1+x^2)^{-4}+(-4)\,48x^3(1+x^2)^{-5}2x)\end{cases}$$

$$=24(1+x^2)^{-3}-6.48x^2(1+x^2)^{-4}+8.48x^4(1+x^2)^{-5}$$

&c. = &c.

Making $x=0$ these several expressions become

$$u=\tan.^{-1}0 \qquad \frac{du}{dx}=1 \qquad \frac{d^2u}{dx^2}=0$$

$$\frac{d^3u}{dx^3}=-2 \qquad \frac{d^4u}{dx^4}=0 \qquad \frac{d^5u}{dx^5}=2^3.3$$

or the developement will be

$$\tan.^{-1}x=0+\frac{x}{1}+0-\frac{2x^3}{2.3}+0+\frac{2.3.4x^5}{2.3.4.5}+0-\&c.$$

$$\text{or } \tan.^{-1}x=\frac{x}{1}-\frac{x^3}{3}+\frac{x^5}{5}-\&c.$$

The series for the *arc* in terms of the *sine*, as well as those for the *sine* and *cosine* in terms of the *arc*, were discovered by Newton before the invention of the general theorems now made use of, in his tract entitled "Analysis per equationes numero terminorum infi- "nitas."

The series for the arc in terms of the *tangent* was discovered by James Gregory in 1671, after a communication of Newton's series for the sine and cosine.

The series in terms of the *sine* is convergent in all cases, since x is always less than unity, and rapidly so when $x=\frac{1}{2}$, or when the arc$=30^\circ$. By means of it Newton calculated the circumference of a circle whose diameter is unity to sixteen places of decimals. This series, with such a calculation to eight places, has been given at the end of the Integral Calculus, (p. 140.) deduced by integration, and having the coefficients in a form which is easily seen to be the same as that here delivered.

(7.) Gregory's formula, $\tan.^{-1} x = x - \frac{x^3}{3} + \frac{x^5}{5} - \&c.$ affords a remarkable expression for the length of the *half quadrant*, and thence of the circumference of a circle. If for x we substitute 1, the tangent of $\frac{\pi}{4}$, we have

$$\frac{\pi}{4} = \frac{1}{1} - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \&c.,$$

a series which is not however sufficiently convergent to be suited for calculation. But it was observed by Machin, that by dividing this arc into several parts whose tangents would be less than unity, we should have converging series for each. He further discovered, that if we take the arc whose tangent is $= \frac{1}{5}$, and that whose tangent is $= \frac{1}{239}$, we shall have

$$\frac{\pi}{4} = 4 \tan.^{-1} \frac{1}{5} - \tan.^{-1} \frac{1}{239}.$$

This is proved as follows: writing $\tan.^{-1} \frac{1}{5} = a$, we have,

$$\tan. 2a = \frac{2 \tan. a}{1 - \tan.^2 a} = \frac{\frac{2}{5}}{1 - \frac{1}{25}} = \frac{5}{12},$$

and thence

$$\tan. 4a = \frac{2 \tan. 2a}{1 - \tan.^2 2a} = \frac{\frac{10}{25}}{1 - \frac{25}{144}} = \frac{120}{119},$$

a value a little greater than unity, or tan. $\frac{\pi}{4}$.

Now therefore subtracting, we have the difference

$$\tan.\left(4a-\frac{\pi}{4}\right)=\tan.\ b=\frac{\tan.\ 4a-\tan.\ \frac{\pi}{4}}{1+\tan.\ 4a\ \tan.\ \frac{\pi}{4}}=\frac{\frac{120}{119}-1}{1+\frac{120}{119}}=\frac{1}{239}.$$

Hence therefore we have the expression

$$\frac{\pi}{4}=4\ \tan.^{-1}\frac{1}{5}-\tan.^{-1}\frac{1}{239}.$$

If we now proceed to develope $\tan.^{-1}\frac{1}{5}$ and $\tan.^{-1}\frac{1}{239}$, we shall have the value of $\frac{\pi}{4}$ by taking a sufficient number of terms of the converging series; or,

$$\frac{\pi}{4}=\left\{\begin{array}{l} 4\left[\frac{1}{5}-\frac{1}{3.5^3}+\frac{1}{5.5^5}-\&c.\right. \\ -\left[\frac{1}{239}-\frac{1}{3(239)^3}+\frac{1}{5(239)^5}-\&c.\right. \end{array}\right\}$$

Euler has investigated other formulæ of the same kind by generalizing the principle, and has applied it to a number of cases. [Novi Comm. Petropol. tom. ix. 1764.

Gregory's series, if we suppose the arc = 30°, and consequently the tangent $=\frac{1}{\sqrt{3}}$ will become

$$30°=\frac{1}{\sqrt{3}}-\frac{1}{3\sqrt{3}.3}+\frac{1}{3^2\sqrt{3}.5}-\&c.$$

Whence

$$3 \,.\, 30^\circ = \frac{\pi}{2} = \sqrt{3} - \frac{1}{3\sqrt{3}} + \frac{1}{3\sqrt{3}\,.\,5} -$$

$$= \sqrt{3}\left(1 - \frac{1}{3\,.\,3} + \frac{1}{3^2.\,5} - \frac{1}{3^3.\,7} + \&c.\right),$$

a form which converges with great rapidity.

(8.) Euler, by a particular artifice, has effected the developement of $\tan.^{-1}(x+h)$ in a very remarkable series, from which some curious and beautiful results are deducible. Let there be assumed

$$u = \tan.^{-1} x = \frac{\pi}{2} - z,$$

$$\text{or} \quad x = \tan.\, u = \cot.\, z.$$

Whence we have $du = dz$; and,

$$\frac{du}{dx} = \frac{1}{1+x^2} = \frac{1}{1+\tan.^2 u} = \frac{1}{\sec.^2 u} = \cos.^2 u = \sin.^2 z = -\frac{dz}{dx}.$$

Differentiating again, we have

$$\frac{d^2u}{dx^2} = \frac{2\sin.\, z \cos.\, z\,.\,dz}{dx} = \frac{dz}{dx}\sin.\, 2z;$$

or, substituting the value of $\frac{dz}{dx}$,

$$\frac{d^2u}{dx^2} = -\sin.^2 z.\, \sin.\, 2z.$$

Again;

$$\frac{d^3u}{dx^3} = \frac{-2\sin.\, z.\, \cos.\, z.\, dz.\, \sin.\, 2z - \sin.^2 \cos.\, 2z.\, 2dz}{dx}$$

$$= -\frac{dz}{dx}\,.\,2\,(\sin.\, z \cos.\, z.\, \sin.\, 2z + \sin.^2 z \cos.\, 2z);$$

$$\text{or, } \frac{d^3u}{dx^3 . 1 . 2}$$

$$= -\frac{dz}{dx}\,[\sin. z]\,[\cos. z \sin. 2z + \sin. z \cos. 2z]$$

$$= -\frac{dz}{dx}\,(\sin. z)\,(\sin. 3z) = \sin.^3 z. \sin. 3z.$$

By repeating the same process, and substituting from the trigonometrical formulæ for the sines of multiple arcs, we obtain successively

$$\frac{d^4u}{1 . 2 . 3 . dx^4} = -\sin.^4 z. \sin. 4z$$

$$\frac{d^5u}{1 . 2 . 3 . 4 . dx^5} = \sin.^5 z \sin. 5z.$$

$$\&c. \qquad = \qquad \&c.$$

Thus upon the whole we have

$$\tan.^{-1}(x+h) = \tan.^{-1}x + \left\{\begin{array}{l} \sin. z. \sin. z. \dfrac{h}{1} \\ - \sin.^2 z \sin. 2z \dfrac{h^2}{2} \\ + \sin.^3 z \sin. 3z \dfrac{h^3}{3} \\ - \sin.^4 z \sin. 4z \dfrac{h^4}{4} \\ + \&c. \end{array}\right.$$

If in this series we make $h = -x$, it gives

$$\tan.^{-1}(0) = \tan.^{-1}x - \sin. z. \sin. z. \frac{x}{1} - \sin.^2 z \sin. 2z \frac{x^2}{2} - \&c.,$$

or substituting $\tan.^{-1}x = \frac{\pi}{2} - z$ and transposing, we have

$$\tan.^{-1}x = \frac{\pi}{2} - z = \sin. z. \sin. z. \frac{x}{1} + \sin.^2 z \sin. 2z. \frac{x^2}{2} + \&c.$$

Since $x=\cot.\, z=\dfrac{\cos.\, z}{\sin.\, z}$, by substituting this value the last series becomes

$$\frac{\pi}{2}-z=\left(\frac{\cos.\, z}{\sin.\, z}\right)\sin.\, z.\ \sin.\, z+\tfrac{1}{2}\left(\frac{\cos.\, z}{\sin.\, z}\right)^2\sin.^2 z\ \sin.\, 2z+\&c.$$

$$\therefore \frac{\pi}{2}=z+\cos.\, z.\ \sin.\, z+\tfrac{1}{2}\cos.^2 z\ \sin.\, 2z+\&c.$$

If, again, we substitute in the first series $-2x$ instead of h, $\tan.^{-1}(x+h)$ becomes $\tan.^{-1}(-x)$. And this by trigonometry $=-\tan.^{-1}x$: hence the series becomes

$$-\tan.^{-1}x=\tan.^{-1}x-\sin.\, z.\ \sin.\, z\,\frac{2x}{1}-\sin.^2 z\ \sin.\, 2z\,\frac{2^2x^2}{2}-\&c.;$$

or transposing

$$2\tan.^{-1}x=\sin.\, z\ \sin.\, z\,\frac{2x}{1}+\sin.^2\ \sin.\, 2z\,\frac{2^2x^2}{2}+\&c.$$

Here again, as before, substituting for x its value $\dfrac{\cos.\, z}{\sin.\, z}$, we deduce (dividing by 2)

$$\frac{\pi}{2}=z+\sin.\, z\ \cos.\, z+\frac{2}{2}\sin.\, 2z\cos.^2 z+\frac{2^2}{3}\sin.\, 3z\ \cos.^3 z+\&c.$$

And if we here make $z=\dfrac{\pi}{4}$, we have

$\sin.\, z=\cos.\, z=\dfrac{1}{\sqrt{2}}$ $\qquad\therefore\sin.\, z\ \cos.\, z=\frac{1}{2}$

$\sin.\, 2z=1\quad \cos.^2 z=\frac{1}{2}$ $\qquad\sin.\, 2z\ \cos.^2 z=\frac{1}{2}$

$\sin.\, 3z=\dfrac{1}{\sqrt{2}}\quad\cos.^3 z=\dfrac{1}{2^{\frac{3}{2}}}$ $\qquad\sin.\, 3z\ \cos.^3 z=\dfrac{1}{2^2}$

$\sin.\, 4z=0$ $\qquad\sin.\, 4z\ \cos.^4 z=0$

$\sin.\, 5z=-\dfrac{1}{\sqrt{2}}\quad\cos.^5 z=\dfrac{1}{2^{\frac{5}{2}}}$ $\qquad\sin.\, 5z\ \cos.^5 z=-\dfrac{1}{2^3}$

&c. $\qquad$ &c.

Continuing in this way to form the successive terms, substituting these values, and transposing $\frac{\pi}{4}$, the last series becomes

$$\frac{\pi}{4}=\frac{1}{2}+\frac{1}{2}+\frac{2^2}{3}\cdot\frac{1}{2^2}-\frac{2^4}{5}\cdot\frac{1}{2^3}-\&c.\,;$$

or multiplying by 2,

$$\frac{\pi}{2}=\frac{1}{1}+\frac{2}{2}+\frac{2}{3}-\frac{2^2}{5}-\frac{2^3}{6}-\frac{2^3}{7}+\frac{2^4}{9}+\&c.$$

These are a few of the various singular deductions made by Euler by means of the series here investigated. Our limits prevent us from entering further upon them: but the student will find them stated in Peacock's Examples, p. 54., and the whole investigation in Euler's Institutiones Calc. Integralis pars ii. art. 57...93.

(9.) The formula for a logarithm in terms of the number or parts of the number (Diff. Calc. form. 26.) becomes, when $x=1$, and $h=u$,

$$\log.\,(1+u)=u-\frac{u^2}{2}+\frac{u^3}{3}-\frac{u^4}{4}+\frac{u^5}{5}\ \&c.$$

And the more convergent series deduced from it will be

$$\log.\left(\frac{1+u}{1-u}\right)=2\left[u+\frac{u^3}{3}+\frac{u^5}{5}+\&c.\right]$$

The former was discovered by Mercator; the latter by James Gregory, (Exercitationes Geom. 1670;) but even the second requires a further transformation to be of much practical utility. This is done by making

$$\frac{1+u}{1-u}=\frac{m}{n} \text{ or } u=\frac{m-n}{m+n};$$

which transforms the above series into

$$\log.\frac{m}{n}=2\left[\frac{m-n}{m+n}+\frac{1}{3}\left(\frac{m-n}{m+n}\right)^3+\&c.\right]$$

And if $m=n+1$, this again becomes

$$\log.(n+1)=\log.n+2\left[\frac{1}{2n+1}+\frac{1}{3}\frac{1}{(2n+1)^3}+\&c.\right]$$

A series which is very convergent, and enables us to calculate the logarithm of any number by means of that which immediately precedes it, or is less by unity.

(10.) The series for finding the value of the modulus of a system of logarithms to a base a, which results from the expression represented by A, (Diff. Calc. p. 12.), by restoring the value of $b=a-1$, or

$$\log.a=A=\left[\frac{a-1}{1}-\frac{(a-1)^2}{2}+\frac{(a-1^3)}{3}-\&c.\right]$$

is not convergent; but a very simple transformation will make it so: since $\log.\sqrt[m]{a}=\frac{1}{m}\log.a$;

and therefore $\log.a=m\log.\sqrt[m]{a}$, we have

$$\log.a=m\left[\frac{\sqrt[m]{a}-1}{1}-\frac{(\sqrt[m]{a}-1)^2}{2}+\&c.\;.\;.\;.\right]$$

And it is evident, that by increasing the value of m this series may be made to converge with any required degree of rapidity.

(11.) It was observed before, (p. 48.), that the formulæ which we have established for the sine and co-

sine in imaginary exponential functions, may also be deduced from the developements of the sine and cosine. We will here shew how, if we know the developement of ϵ^x, and consequently that of $\epsilon^{x\sqrt{-1}}$ and of $\epsilon^{-x\sqrt{-1}}$, we can from these formulæ obtain the developement of the sine and cosine.

From the developement of ϵ^x we have

$$\epsilon^{x\sqrt{-1}} = 1 + x\sqrt{-1} + \frac{x^2(-1)}{1\,.\,2} + \frac{x^3(-1)^{\frac{3}{2}}}{1\,.\,2\,.\,3} + \&c.$$

$$= 1 + x\sqrt{-1} - \frac{x^2}{1\,.\,2} - \frac{x^3}{1\,.\,2\,.\,3} + \frac{x^4}{1\,.\,2\,.\,3\,.\,4} +$$

$$\epsilon^{-x\sqrt{-1}} = 1 - x\sqrt{-1} - \frac{x^2}{1\,.\,2} + \frac{x^3}{1\,.\,2\,.\,3} + \frac{x^4}{1\,.\,2\,.\,3\,.\,4} -$$

$$\therefore\ \epsilon^{x\sqrt{-1}} + \epsilon^{-x\sqrt{-1}} = 2 - \frac{2x^2}{1\,.\,2} + \frac{2x^4}{1\,.\,2\,.\,3\,.\,4} - \&c.$$

or dividing by 2, we find

$$\cos.\, x = 1 + \frac{x^2}{1\,.\,2} + \frac{x^4}{1\,.\,2\,.\,3\,.\,4} - \&c.$$

And in a precisely similar manner we get the developement of the sine.

Again, from the expression

$$\epsilon^{x\sqrt{-1}} = \cos.\, x + \sqrt{-1}\ \sin.\, x,$$

which by trigonometry $= \cos.\, x\ (1 + \sqrt{-1}\ \tan.\, x)$;

by taking the logarithms on each side, and observing that $\log.\, \epsilon = 1$, and therefore

$$\log.\, \epsilon^{x\sqrt{-1}} = x\sqrt{-1}\ (1);$$

we have

$$x\sqrt{-1} = \log.\, \cos.\, x + \log.\, (1 + \sqrt{-1}\ \tan.\, x).$$

And the developement of the second member by the formula for log. $(1+h)$ gives

$$x\sqrt{-1} = \log.\cos.x + \sqrt{-1}\tan.x + \left.\begin{array}{l} \frac{\tan.^2 x}{2} - \frac{\sqrt{-1}\tan.^3 x}{3} \\ \frac{-\tan.^4 x}{4} + \&c. \end{array}\right\}$$

But since the imaginary and the real terms on each side must be equal, we have

$$x\sqrt{-1} = \sqrt{-1}\left(\tan.x - \frac{\tan.^3 x}{3} + \frac{\tan.^5 x}{5} - \&c.\right)$$

and therefore

$$x = \tan.x - \frac{\tan.^3 x}{3} + \frac{\tan.^5 x}{5} - \&c.$$

(12.) If in the series, for log. $(x+h)$, [Diff. Calc. form (26,)] we make $x=1$, it becomes

$$\log.(1+h) = 0 + h - \frac{h^2}{2} + \frac{h^3}{3} - \&c.$$

Again, if we make $h=1$, we have

$$\log.(x+1) = \log.x + \frac{1}{x} - \frac{1}{2x^2} + \frac{1}{3x^3} - \&c.$$

Hence, transposing log. x, and subtracting the second series from the first, we obtain

$$\log.\left(\frac{1+h}{x+1}\right) + \log.x = h - \frac{h^2}{2} + \frac{h^3}{3} - \&c. - \left(\frac{1}{x} - \frac{1}{2x^2} + \frac{1}{3x^3} - \right)$$

But since x and h are both arbitrary, we may suppose them equal, and with them $=u$; in which case the first term becomes log. $1=0$, and the series may be expressed in the form

$$\log.u = (u - u^{-1}) - \frac{(u^2 - u^{-2})}{2} + \frac{(u^3 - u^{-3})}{3}$$

This expression is remarkable for the peculiarity of its form: it has also one or two somewhat singular applications, which we will proceed to explain.

If for u we write $\sqrt{-1}$, the series will become

$$\log.\sqrt{-1} = \left[\sqrt{-1} - \frac{1}{\sqrt{-1}}\right] - \frac{1}{2}\left[(-1) - \frac{1}{(-1)}\right]$$

$$+\frac{1}{3}\left[(-1)^{\frac{3}{2}} - \frac{1}{(-1)^{\frac{3}{2}}}\right] - \frac{1}{4}\left[(-1)^{2} - \frac{1}{(-1)^{2}}\right] + \&c.$$

which, reducing each of the terms to common denominators, becomes

$$= \left[\frac{-1-1}{\sqrt{-1}}\right] - \frac{1}{2}\left[\frac{1-1}{-1}\right] + \frac{1}{3}\left[\frac{-1-1}{(-1)\sqrt{-1}}\right] - \frac{1}{4}\left[\frac{1-1}{-1}\right] + \&c.$$

or observing that the alternate terms vanish, we have

$$\log.\sqrt{-1} = \frac{-2}{\sqrt{-1}}\left[1 - \frac{1}{3} + \frac{1}{5} - \&c.\right]$$

But the last factor is the series we have already found for the value of $\frac{\pi}{4}$: hence,

$$\log.\sqrt{-1} = \frac{-1}{\sqrt{-1}} \cdot \frac{\pi}{2};$$

$$\text{or } (\log.\sqrt{-1})\sqrt{-1} = -\frac{\pi}{2}$$

and consequently

$$(\sqrt{-1})^{\sqrt{-1}} = \epsilon^{-\frac{\pi}{2}}.$$

This remarkable expression was discovered by John Bernouilli.

It may also be deduced from the imaginary expression for the trigonometrical lines.

(13.) The above formula for log. u has also another remarkable application to trigonometrical functions.

For if we write $\epsilon^{x\sqrt{-1}}$, in the place of u, and divide the whole by $2\sqrt{-1}$, observing that on this supposition log. u becomes log. $\epsilon^{x\sqrt{-1}} = x\sqrt{-1}$, we have

$$\frac{x}{2} = \frac{\epsilon^{x\sqrt{-1}} - \epsilon^{-x\sqrt{-1}}}{2\sqrt{-1}} - \frac{1}{2}\,\frac{\epsilon^{2x\sqrt{-1}} - \epsilon^{-2x\sqrt{-1}}}{2\sqrt{-1}} + \&c.$$

$$\text{or } \frac{x}{2} = \sin.\,x - \frac{\sin.\,2x}{2} + \frac{\sin.\,3x}{3} - \&c.$$

An expression for x in terms of the sines of x and its multiples, first given by Euler.

Again, if we differentiate successively this equation, we shall have the following curious results:

$$\tfrac{1}{2} = \cos.\,x - \cos.\,2x + \cos.\,3x - \&c.$$

$$0 = -\sin.\,x + 2\sin.\,2x - 3\sin.\,3x + \&c.$$

$$0 = -\cos.\,x + 2^2.\cos.\,2x - 3^2\cos.\,3x + \&c.$$

and proceeding in this way, we have generally

$$0 = \cos.\,x - 2^{2n}\cos.\,2x + 3^{2n}\cos.\,3x - \&c.$$

$$0 = \sin.\,x - 2^{2n+1}\sin.\,2x + 3^{2n+1}\sin.\,3x - \&c.$$

If in the first of these expressions we make $x = 0$, we have

$$0 = 1 - 2^{2n} + 3^{2n} - 4^{2n} + \&c.$$

and if in the second we make $x = \frac{\pi}{2}$, it gives

$$0 = 1 - 3^{2n+1} + 5^{2n+1} - 7^{2n+1} + \&c.$$

THE THEOREMS OF LAGRANGE AND LAPLACE.

(14.) The most comprehensive and important formulæ for the developement of functions remain to be investigated. We will suppose a function of a very general form

$$u = fy, \text{ where } y = z + x\phi y,$$

and let z be supposed independent of the variation of x. Then, in order to obtain the developement of

$$u = f[z + x\phi y],$$

we shall first treat $x\phi y$ as arbitrary, and develope the function by Taylor's theorem, which gives

$$u = fz + \frac{dfz}{dz} x\phi y + \frac{d^2fz}{dz^2} \frac{x^2\phi y^2}{1 \cdot 2} + \&c.$$

$$\text{But } \phi y = \phi\, [z + x\phi y].$$

This again therefore may be developed in the same way, or

$$\phi y = \phi z + \frac{d\phi z}{dz} x\phi y + \frac{d^2\phi z}{dz^2} \frac{x^2\phi y^2}{1 \cdot 2} + \&c.$$

We have then to substitute this series, its square, &c. in the successive terms of the former. But again, since ϕy enters into the several terms of this second series, we must in each of these again make the same substitution, and so on to an indefinite extent. The

series which results from these continual substitutions will therefore be formed in the following manner:

$$u = \left\{ \begin{array}{l} fz + \dfrac{dfz}{dz}x \left\{ \begin{array}{l} \phi z + \dfrac{d\phi z}{dz}\, x \,[\phi z + \dfrac{d\phi z}{dz}x\,(\phi z + \&c.)] \\ \qquad + \dfrac{d^2\phi z}{dz^2}\, \dfrac{x^2}{1\,.\,2}\,[\phi z + \&c.]^2 + \&c. \end{array} \right\} \\ + \dfrac{d^2fz}{dz^2}\, \dfrac{x^2}{1\,.\,2} \left\{ \begin{array}{l} \phi z + \dfrac{d\phi z}{dz}\, x\,(\phi z + \&c.) \\ \quad + \&c.\ \&c. \end{array} \right\}^2 \\ + \dfrac{d^3fz}{dz^3}\, \dfrac{x^3}{1\,.\,2\,.\,3} \left\{ \begin{array}{l} \phi z + \dfrac{d\phi z}{dz}\, x\,(\phi z + \&c.) \\ \quad + \&c.\ \&c. \end{array} \right\}^3 \\ + \&c.\ \&c. \quad .\ .\ .\ .\ .\ .\ .\ .\ .\ . \end{array} \right. \qquad (1)$$

It only remains to arrange this developement according to the powers of x. The first power of x is evidently contained in only one term.

We also readily find that the terms involving x^2 are,

$$\left(\frac{dfz}{dz} \cdot \frac{d\phi z}{dz} \cdot \phi z + \frac{d^2fz}{dz^2}\frac{\phi z}{1\,.\,2}\right)x^2.$$

Again, observing that the expansions of the series which are to be raised to the second power, will give,

$$\phi z^2 + 2\phi z\frac{d\phi z}{dz}x(\phi z + \&c.) + \&c.$$

We shall be easily able to collect the coefficients of x^3; which will be

$$\left. \begin{array}{l} \left(\dfrac{dfz}{dz} \cdot \dfrac{d\phi z}{dz} \cdot \dfrac{d\phi z}{dz} \cdot \phi z\right) \\ + \left(\dfrac{dfz}{dz} \cdot \dfrac{d^2\phi z}{dz^2} \cdot \dfrac{\phi z^2}{1\,.\,2}\right) \\ + 2\left(\dfrac{d^2fz}{dz^2} \cdot \dfrac{d\phi z}{dz} \cdot \dfrac{\phi z^2}{1\,.\,2}\right) \\ + \left(\dfrac{d^3fz}{dz^3} \cdot \dfrac{\phi z^3}{1\,.\,2\,.\,3}\right) \end{array} \right\} x^3$$

$$\text{Or, }\left.\begin{array}{l} 1\,.\,2\,.\,3\,.\,dfz\,.\,(d\phi z)\,.\,^2\phi z \\ +\,3\,.\,dfz\,.\,d^2\phi z\,.\,\phi z^2 \\ +\,2\,.\,3\,.\,d^2fz\,.\,d\phi z\,.\,\phi z^2 \\ +\qquad d^3fz\,.\,\phi z^3 \end{array}\right\}\frac{1}{dz^3}\cdot\frac{x^3}{1\,.\,2\,.\,3}$$

In like manner we may collect the terms involving x^4, &c.

But, upon examining these coefficients, we find that the coefficient of x^2 may be written in the form

$$\frac{d\left(\dfrac{dfz}{dz}\right)\phi z^2+\dfrac{dfz}{dz}\,.\,2\phi z\,.\,d\phi z,}{dz}$$

which is immediately seen to be the differential

$$\frac{d\left(\dfrac{dfz}{dz}(\phi z)^2\right)}{dz}$$

Again, the coefficient of x^3 being written

$$\left.\begin{array}{l} 2\,.\,3\,.\,\phi z\,.\,d\phi z\,.\,d\phi z\,.\,dfz \\ +\,3\,.\,\phi z^2\,.\,d^2\phi z\,.\,dfz \\ +\,3\;\;\phi z^2\,.\,d\phi z\,.\,d^2fz \\ +\,3\;\;\phi z^2\,.\,d\phi z\,.\,d^2fz \\ +\quad\phi z^3\,.\,d^3fz \end{array}\right\}\frac{1}{dz^3},$$

is easily seen to be the quantity which results from taking the differential.

$$d\left(\frac{3\phi z^2\,.\,d\phi z\,.\,dfz}{dz^3}+\frac{\phi z^3\,.\,d^2fz}{dz^3}\right)$$

$$\text{or,}\quad\frac{d^2\left(\phi z^3\,\dfrac{dfz}{dz}\right)}{dz^2}$$

And similarly the values of the other coefficients may be found.

Substituting these values then in the form (1), we shall, upon the whole, obtain the developement,

$$\left.\begin{aligned} u = fz + \frac{dfz}{dz}\phi z \cdot \frac{x}{1} + \frac{d\left(\frac{dfz}{dz}(\phi z)^2\right)}{dz} \frac{x^2}{1.2} \\ + \frac{d^2\left(\frac{dfz}{dz}(\phi z)^3\right)}{dz^2} \cdot \frac{x^3}{1.2.3} \\ + \&c. \end{aligned}\right\} \ldots (2.)$$

(15.) This formula was first given by Lagrange in the Mémoires de l'Académie de Berlin, 1768; and it has been since usually known by the name of Lagrange's Theorem. It obviously includes that of Taylor, to which it is reduced, if we suppose $x\phi y$ to become constant, and $= h$.

But this formula is itself susceptible of being made yet more general, if, instead of supposing $y = z + \phi y$, we suppose it *any function* of that quantity, or, $y = f(z + \phi y)$; we shall then have,

$$u = f(y) = f[f(z + x\phi(y))],$$

but this may be written more simply

$$= \psi[z + x\phi(y)].$$

And under this form we can apply directly the preceding process, or we have first,

$$u = \psi z + \frac{d\psi z}{dz} x\phi y + \frac{d^2\psi z}{dz^2} \frac{x^2(\phi y)^2}{1.2} + \&c.$$

and then assuming

$$\phi y = \phi[f(z + x\phi y)] = \theta(z + x\phi y)$$

$$= \theta z + \frac{d\theta z}{dz} x\phi y + \&c.$$

and substituting as before, we ultimately obtain

$$u = \psi z + \theta z \frac{d\psi z}{dz} \cdot x + d \cdot \frac{(\theta z)^2 \frac{d\psi z}{dz}}{dz} \quad \frac{x^2}{1.2} + \&c. \quad . \; . \; . \; (3.)$$

This more general formula was given by Laplace in the Mém. de l'Acad. des Sciences, 1777, and is commonly called Laplace's Theorem.

The following are two modifications in the form of Lagrange's series, which are sometimes useful:

If we have $x = 1$, the formula becomes

$$u = fz + \phi z \frac{dfz}{dz} + \frac{d\left((\phi z)^2 \frac{dfz}{dz}\right)}{dz} \quad \frac{1}{1.2} + \&c. \quad . \; . \; . \; . \; (4.)$$

If we have $fz = z$, and therefore, $\frac{dfz}{dz} = 1$, it will become,

$$u = z + \phi z x + \frac{d\phi z^2}{dz} \cdot \frac{x^2}{1.2} + \&c. \quad . \; . \; . \; . \; . \; . \; . \; (5.)$$

(16.) The formulæ of Lagrange and Laplace are applicable to the developement of all functions *explicit or implicit.* Of the former it will not be necessary here to adduce examples: of the latter we shall give some instances.

Let the given implicit function be that expressed by the equation

$$y^3 - 3axy + x^3 = 0.$$

In order to compare this with the formula,

$$y - (z + (x)\phi y) = 0,$$

we must write it in the form

$$y - \left(\frac{x^2}{3a} + \frac{y^3}{3ax}\right) = 0.$$

Whence we find

$$\mathfrak{z}=\frac{x^2}{3a}, \qquad x\phi y=\frac{1}{3ax}\,.\,y^3,$$

$$\text{or,}\,(x)=\frac{1}{3ax}, \text{ and } \phi y=y^3\,; \text{ whence } \phi\mathfrak{z}=\mathfrak{z}^3\,;$$

Also it is proposed to find $fy=y$, $\therefore f\mathfrak{z}=\mathfrak{z}$, or we shall use the form (5), in which the successive coefficients are easily formed; thus

$$\frac{d\,\phi\mathfrak{z}^2}{d\mathfrak{z}}=\frac{d\,\mathfrak{z}^6}{d\mathfrak{z}}=6\mathfrak{z}^5=6\,.\,\frac{x^{10}}{3^5\,a^5},$$

and so on. Thus on the whole we find the developement,

$$y=\frac{x^2}{3a}+\frac{x^6}{3^3\,a^3}\,.\,\frac{1}{3ax}+\frac{6\,x^{10}}{3^5\,a^5}\,.\,\frac{1}{3^2\,a^2\,x^2}\,.\,\frac{1}{1\,.\,2}+\&\text{c.}$$

$$=\frac{x^2}{3a}+\frac{x^5}{3^4\,a^4}+\frac{6\,x^8}{3^7\,a^7\,1\,.\,2}+\&\text{c.}$$

(17.) We will not however dwell upon the cases of particular equations, but proceed to a simple instance of an equation of m dimensions, and shew how we can obtain *an expression in a series for any power of the variable*. Thus let it be required to find an expression for y^n when we have given the equation of the *mth* degree,

$$\alpha-\beta y+\gamma y^m=0.$$

Transposing and dividing by β, this gives

$$y=\frac{\gamma}{\beta}\,y^m+\frac{\alpha}{\beta}\,;$$

and comparing this with the form assumed for Lagrange's Theorem,

$$y = z + x\phi y,$$

we have $\quad z = \frac{\alpha}{\beta}$

$$x\phi y = \frac{\gamma}{\beta} y^m \quad \therefore x = \frac{\gamma}{\beta}, \qquad \phi y = y^m.$$

The required form of fy is y^n; whence $fz = z^n = \frac{\alpha^n}{\beta^n}$,

and $\phi z = z^m = \frac{\alpha^m}{\beta^m}$.

From these data we proceed to form the coefficients:

$$\frac{dfz}{dz} = n\, z^{n-1} = n\,\frac{\alpha^{n-1}}{\beta^{n-1}}.$$

$$\frac{d\left(\frac{dfz}{dz}\,\phi z^2\right)}{dz} = \frac{d\,[nz^{n-1}\, z^{2m}]}{dz}$$

$$= n\,(2m+n-1)\; z^{2m+n-2}$$

$$= n\,(2m+n-1)\,\frac{\alpha^{2m+n-2}}{\beta^{2m+n-2}}$$

$$\frac{d^2\left(\frac{dfz}{dz}\,\phi z^3\right)}{dz} = \frac{d^2\,[nz^{n-1}\, z^{3m}]}{dz}$$

$$= d\,\frac{[n\,(3m+n-1)\; z^{3m+n-2}}{dz}$$

$$= n\,(3m+n-1)\;(3m+n \quad 2)\; z^{3m+n-3}$$

$$= n\,(3m+n-1)\,(3m+n-2)\,\frac{\alpha^{3m+n-3}}{\beta^{3m+n-3}}.$$

And similarly for the coefficients of the other terms: thus upon the whole the developement will be

$$y^n = \frac{\alpha^n}{\beta^n} + n.\,\frac{\alpha^{n-1}}{\beta^{n-1}}\,\frac{\alpha^m\gamma}{\beta^{m+1}} + n(2m+n-1)\frac{\alpha^{2m+n-2}}{\beta^{2m+n-2}}.\frac{\gamma^2}{\beta^2}.\frac{1}{1\,.\,2} + \&c.$$

$$= \frac{\alpha^n}{\beta^n}\left(1 + n\,\frac{\alpha^{m-1}}{\beta^m}\gamma + \frac{n(2m+n-1)}{1\,.\,2}\,\frac{\alpha^{2m-2}\gamma^2}{\beta^{2m}} + \&c.\right)\ldots (6).$$

If we wished to express the value of the first power of y, we have only in this form to substitute $n=1$, and it gives

$$y=\frac{\alpha}{\beta}\left(1+\frac{\alpha^{m-1}}{\beta^m}\gamma+\frac{2m}{1.2}\frac{\alpha^{2m-2}}{\beta^{2m}}\gamma^2+\frac{3m(3m-1)}{1.2.3}\frac{\alpha^{3m-3}}{\beta^{3m}}\gamma^3+\&c.\right)(7).$$

The same general formula will also give us another expression for y, by a transformation of the original equation.

If in the given equation we write $y^m=u$, it becomes

$$\alpha-\beta u^{\frac{1}{m}}+\gamma u=0.$$

Or if to preserve the analogy of the notation we write

$$\gamma=-b, \qquad \beta=-c,$$

it becomes

$$a-bu+cu^{\frac{1}{m}}=0$$

Then applying to this equation the preceding developement, by making $(n)=(m)=\frac{1}{m}$, the form for

$$(y^n)=u^{\frac{1}{m}} \text{ becomes}$$

$$u^{\frac{1}{m}}=\left(\frac{a}{b}\right)^{\frac{1}{m}}\left[1+\frac{1}{m}\frac{a^{\frac{1}{m}-1}c}{b^{\frac{1}{m}}}+\frac{\frac{1}{m}\left(\frac{2}{m}+\frac{1}{m}-1\right)}{1.2}\frac{a^{\frac{2}{m}-2}c^2}{b^{\frac{2}{m}}}+\&c.\right]$$

$$=\left(\frac{a}{b}\right)^{\frac{1}{m}}\left(1+\frac{1}{m}\frac{a^{\frac{1}{m}}}{b^{\frac{1}{m}}}\frac{c}{a}+\frac{3-m}{m^2.1.2}\frac{a^{\frac{2}{m}}}{b^{\frac{2}{m}}}\frac{c^2}{a^2}+\&c.\right)$$

Or if we write

$$\left(\frac{a}{b}\right)^{\frac{1}{m}}=\left(\frac{\alpha}{-\gamma}\right)^{\frac{1}{m}}=\delta,$$

and restore the value of $c=-\beta$, this becomes

$$u^{\frac{1}{m}} = y = \delta\left(1 - \frac{1}{m}\,\frac{\delta\beta}{\alpha} + \frac{3-m}{m^2.1.2}\,\frac{\delta^2\beta^2}{\alpha^2} - \frac{(4-m))4-2m)}{m^3.1.2.3}\right.$$
$$\left.\frac{\delta^3\beta^3}{\alpha^3} + \&c.\right)\ldots.(8).$$

The developement of the value of y in this form affords us the important conclusion, that, *as the quantity which we have designated by* δ, or $\left(\frac{\alpha}{-\gamma}\right)^{\frac{1}{m}}$ *is susceptible of* m *values, the expression for* y *has as many, which are, in other words, the* m *roots of the given equation.* These values of δ are expressed by the general form

$$\left(\cos.\frac{2\pi n}{m} + \sqrt{-1}\,\sin.\frac{2\pi n}{m}\right)\left(\frac{\alpha}{-\gamma}\right)^{\frac{1}{m}}.$$

and the particular values are obtained by substituting successively the numbers

$$1\,,\ 2\,,\ 3\ \ldots .\ m, \text{ for } n.$$

Lagrange has shewn that the value of y, as given by the first series, is that of the *least* root of the equation. He has also investigated many curious properties of the roots arising out of these developements. The form of the equation we have here considered is a very limited case. Lagrange has applied the investigation to an equation of m dimensions in its most general form; and has further shewn the conditions for ascertaining the degree of convergency of the series in different cases. But for an account of these points the student is referred to Lagrange's Memoirs in the Mém. Acad. de Berlin 1768 and 1770. Also to his Traité de la Résolution des Equations numériques: note xi.

(18.) We shall here only proceed to illustrate the process by applying it to the cubic equation,

$$q - py + y^3 = 0.$$

Here we have to substitute in the series for y, form (7),

$$\frac{\alpha}{\beta} = \frac{q}{p} \qquad \gamma = 1 \qquad m = 3,$$

and it becomes

$$y = \frac{q}{p}\left(1 + \frac{q^2}{p^3} + \frac{6}{1.2}\,\frac{q^4}{p^6} + \frac{8.9}{1.2.3}\,\frac{q^6}{p^9} + \&c.\right)$$

Again, in the second series, form (8), the substitutions will be

$$\alpha = q \quad \beta = p \quad \delta = \left(\frac{\alpha}{-\gamma}\right)^{\frac{1}{m}} = (-q)^{\frac{1}{3}},$$

and we shall have the developement,

$$u^{\frac{1}{3}} = y = (-q)^{\frac{1}{3}}\left\{\begin{array}{l} 1 - \frac{1}{3}\,\frac{p(-q)^{\frac{1}{3}}}{q} + \frac{3-3}{3^2.1.2}\,\frac{p^2(-q)^{\frac{2}{3}}}{q^2} \\ \qquad - \frac{(1)\,(-2)}{3^3.1.2.3}\,\frac{p^3(-q)}{q^3} \\ \qquad + \&c. \end{array}\right\}$$

$$= (-q)^{\frac{1}{3}}\left(1 + \frac{1}{3}\,\frac{p}{(-q)^{\frac{2}{3}}} + 0 + \frac{1(-2)}{3^3.1.2.3}\,\frac{p^3}{(-q)^2} + \&c.\right)$$

Here it is easily seen that $(-q)^{\frac{1}{3}}$ admits of three values, which are

$$(-q)^{\frac{1}{3}} \qquad \left(\frac{1-\sqrt{-3}}{2}\right)q^{\frac{1}{3}} \qquad \left(\frac{1+\sqrt{-3}}{2}\right)q^{\frac{1}{3}}.$$

Since the product of these factors will be

$$\left(\frac{1-(-3)}{4}\right)q^{\frac{2}{3}}(-q)^{\frac{1}{3}} = -q.$$

The series for y therefore has three values, according as we substitute each of these values of $(-q)^{\frac{1}{3}}$, and thus we have developements for each of the three roots of the cubic equation.

(19.) Another important case of these applications is that which refers to *the reversion of series.* For example, if we have given the series

$$\alpha + \beta y + \gamma y^2 + \delta y^3 + \&c.$$

and it be required to revert it, or to express y in a series, we must first divide by β, and then transposing, we have

$$y = -\frac{\alpha}{\beta} - \frac{1}{\beta}(\gamma y^2 + \delta y^3 + \&c.)$$

which may be compared with the form

$$u = f(z + x\phi y),$$

and we find $z = -\frac{\alpha}{\beta}$

$$x\phi y = -\frac{y^2}{\beta}(\gamma + \delta y + \&c.)$$

whence $x = 1$, also $fy = y$, and therefore $fz = z$:

$$\text{also } \phi z = -\frac{z^2}{\beta}(\gamma + \delta z + \&c.)$$

Thus the conditions both of formula (4) and (5) are fulfilled, or we have the developement

$$y = z - \frac{z^2}{\beta}(\gamma + \delta z + \&c.) + \frac{1}{1.2.dz} . d\left(\frac{z^4}{\beta^2}(\gamma + \delta z + \&c.)^2\right)$$
$$- \frac{1}{1.2.3.dz^2} d^2\left(\frac{z^6}{\beta^4}(\gamma + \delta z + \&c.)^3\right)$$
$$+ \&c.$$

Expanding and performing the differentiations indicated in the coefficients, they become respectively

$$d\left(\frac{z^4}{\beta^2}\right)(\gamma^2+2\gamma\delta z+\&c.)$$

$$=dz\left(\frac{4\gamma^2z^3}{\beta^2}+\frac{2.5.\gamma\delta z^4}{\beta^2}+\&c.\right)$$

$$\text{and } d^2\left(\frac{z^6}{\beta^4}\right)(\gamma^3+3\gamma^2\delta z+\&c.)$$

$$=d\left(\frac{6\gamma^3z^5dz}{\beta^4}+\&c.\right)$$

$$=dz^2\left(\frac{5.6.\gamma^3z^4}{\beta^4}+\&c.\right)$$

And we thus have the developement

$$\left.\begin{array}{r} y=z-\dfrac{\gamma}{\beta}z^2-\dfrac{\delta}{\beta}z^3-\dfrac{\epsilon}{\beta}z^4-\&c. \\ +\dfrac{2\gamma^2}{\beta^2}z^3+\dfrac{5\gamma\delta}{\beta^2}z^4+\&c. \\ -\dfrac{5\gamma^3}{\beta^4}z^4-\&c. \end{array}\right\}$$

(20.) We will only add two further applications of Lagrange's formula:

Let it be required to develope the function

$$u=nt+e\sin. u.$$

Comparing this with the expression

$$u=f(z+x\phi y),$$

we have

$$nt=z \quad x=e \quad y=u \quad \phi y=\sin. u$$
$$fz=z \quad \phi z=\sin. z.$$

We will first point out the mode of forming the successive differentials

$$d\,(\phi z)^2, \quad d^2\,(\phi z)^3 \text{ \&c.}$$

$$\text{or } d\,.\,\sin.^2 z, \quad d^2 \sin.^3 z \text{ \&c.}$$

For the first we have evidently

$$\frac{d \sin.^2 z}{dz} = 2 \sin.\, z \cos.\, z$$
$$= \sin.\, 2\, z.$$

To obtain the second, we have first

$$\frac{d \sin.^3 z}{dz} = 3 \sin.^2 z \cos.\, z\,;$$

and thence

$$\frac{d\,.^2 \sin.^3 z}{dz^2} = 3\,.\,2 \sin.\, z\,.\,\cos.\, z\,.\,\cos.\, z + 3 \sin.^2 z\,(-\sin. z)$$
$$= 3\,.\,2\,.\,\sin.\, z \cos.^2 z - 3 \sin.^3 z.$$
$$= 3\,(2 \sin.\, z \cos.^2 z - \sin.^3 z)$$
$$= 3\,(2 \sin.\, z\,(1 - \sin.^2 z) - \sin.^3 z)$$
$$= 3\,(2 \sin.\, z - 3 \sin.^3 z).$$

But by the formula (p. 53.) making $m = 3$, we have

$$\sin.^3 z = \frac{1}{2\,.\,2\,.^2(-1)} \left\{ \begin{array}{r} \sin. 3z - 3 \sin. z + 3 \sin.\,(-1)\, z \\ - \sin.\,(-3)\, z) \end{array} \right\}$$
$$= -\frac{1}{2\,.\,2^2}\,(2 \sin.\, 3\, z + 2\,.\,3 \sin.\, z).$$

Hence the differential becomes

$$\frac{d^2 \sin.^3 z}{dz^2} = 3 \left\{ 2 \sin. z - 3 \left(\frac{1}{-2.2^2}\,(\sin.\, 3\, z + 3 \sin.\, z) \right) \right\}$$
$$= 3 \left\{ \frac{3}{2^2} \sin.\, 3\, z - \frac{3\,.\,3}{2^2} \sin.\, z + 2 \sin.\, z \right\}$$

$$=3\left\{\frac{3}{2^2}\sin.3z+\left(\frac{2.2^2-3.3}{2^2}\right)\sin.z\right\}$$

$$=\frac{3}{2^2}\left\{3\sin.3z-\sin.z\right\}$$

We must proceed in like manner to form the other differential coefficients: and thus upon the whole we shall obtain the developement which comes under the case of form (5).

$$u=z+e\sin.z+\frac{e^2}{1.2}\frac{d(\sin.^2z)}{dz}+\frac{e^3}{1.2.3}.\frac{d^2(\sin.^3z)}{dz}+\&c.$$

which becomes, by substituting the values above obtained,

$$u=\begin{cases} nt+e\sin.nt+\dfrac{e^2}{1.2.}\sin.2nt \\ +\dfrac{e^3}{1.2.3}.\dfrac{1}{2^2}(3^2\sin.3nt-3\sin.nt) \\ +\ \&c.\ \&c. \end{cases}$$

When e is very small, this series converges with great rapidity. It is a developement of considerable importance in astronomy: and in the practical case to which it is applied, the condition, that e is very small, is fulfilled.

This developement is closely connected with another in which we have to express a function of u, (the quantity above represented,) which is given in the form,

$$fu=r=a(1-e\cos.u).$$

In order to do this, we must observe that in the general form of the series, (writing y for u to preserve the notation of the last article,)

$$y=\psi u=\psi z+\left(\phi z\frac{d\psi z}{dz}\right)\frac{x}{1}+d\frac{\left(\phi z^2\frac{d\psi z}{dz}\right)}{dz}\frac{x^2}{1.2}+\&c.$$

if we substitute

$$\psi z = \psi nt,\ x = e, \text{ and } \phi z = \sin. z = \sin. nt,$$

it becomes

$$\psi u = \psi nt + e\left(\sin.\ nt\,\frac{d\psi nt}{dnt}\right) + \&c.$$

and if we suppose the function $\psi u = \cos. u$, then likewise $\psi nt = \cos. nt$; and

$$\cos.u = \cos.nt + e\sin.nt.\frac{d\cos.nt}{dnt} + \frac{e^2}{1.2}\frac{d\left(\sin.^2 nt.\frac{d\cos.nt}{dnt}\right)}{dnt} + \&c.$$

$$= \cos.\ nt - e\sin.^2 nt - \frac{e^2}{1.2}\,\frac{d\sin.^3 nt}{ndt} + \&c.$$

If we now perform the differentiations, and substitute their values as in the preceding process, and substitute this developement in the equation

$$r = a\,(1 - e\cos.\ u),$$

it becomes

$$r = a\left[1 - e\cos.\ nt + e^2(1 - \cos.^2 nt\right] + \frac{e^3}{1.2}\left[3\sin.^2 nt\cos.nt\right] + \&c.$$

But developing the powers of cos. nt by formula (11. p. 51,) we have

$$1 - \cos.^2 nt = 1 - \tfrac{1}{2}\cos.\ 2nt + \tfrac{1}{2} = \frac{1 - \cos.\ 2nt}{2}$$

$$\left[3 - 3\cos.^2 nt\right]\cos.\ nt = 3\cos.\ nt - 3\cos.^3 nt$$

$$= 3\cos.nt - \left(\frac{3}{4}\cos.3nt + \frac{3.3}{4}\cos.\ nt\right)$$

$$= -\frac{3}{4}\cos.\ 3nt + \frac{3}{4}\cos.\ nt;$$

and proceeding in this way we shall obtain the developement

$$r = a\Big[1 - e\cos. nt + \frac{e^2}{2}(1 - \cos.2\,nt) - \frac{e^3}{1\,.\,2\,.\,2^2}(3\cos.3\,nt - 3\cos.\,nt) - \&c.\Big]$$

The application of these developements will be seen in Woodhouse's Astronomy, vol. II. ch. xviii. and Pontecoulant, Systeme du Monde, tom. I. p. 261.

INTEGRAL CALCULUS.

IN this division of the subject we propose to continue the same plan as has been followed in illustrating the Differential Calculus. A few examples, carefully selected and fully developed, will afford sufficient explanation of the process of integration, where it can be reduced to general principles; as well as some exemplification of those artifices, the application of which depends on the skill of the analyst, and can only be learnt by experience.

ELEMENTARY INTEGRATIONS.

The following are a few examples of the integration of elementary differentials, and of such expressions as are directly reducible to elementary forms.

1.] To integrate $\left(adx - \frac{bdx}{x^3} + xdx\sqrt{x}\right)$

we have to take

$$\int adx - \int \frac{bdx}{x^3} + \int x^3 dx.$$

But we find 1st, $\int adx = ax + C_1$,

$$\text{2dly, } \int \frac{bdx}{x^3} = b \cdot \int \frac{dx}{x^3} = \frac{b}{2} \int \frac{2xdx}{x^4},$$

under which form it is immediately seen to be a simple differential, or we find it by Int. Calc. (4)

$$= \frac{b}{2} \cdot \left(\frac{-1}{x^2}\right) + C_2.$$

3dly. For the remaining term we have by form (6),

$$\int x^{\frac{3}{2}} dx = \frac{x^{\frac{3}{2}+1}}{\frac{3}{2}+1} + C_3$$

$$= \frac{2x^{\frac{5}{2}}}{5} + C_3.$$

Let $C_1 + C_2 + C_3 = C$,

thence the whole integral will be

$$ax + \frac{b}{2x^2} + \frac{2}{5} x^{\frac{5}{2}} + C.$$

2.] To integrate

$$du = \left(\frac{mdx}{x} + \frac{ndy}{y} + \frac{rdz}{z}\right) x^m y^n z^r,$$

we perceive immediately on multiplying, that the in-integral will be

$$u = x^m y^n z^r + C.$$

3.] To integrate

$$du = \frac{xdx}{\sqrt{x^2 + a^2}},$$

we see immediately, on the principles of differentiation, that the integral will be

$$u = \sqrt{a^2 + x^2} + C.$$

In the same way we find

$$\int \frac{x dx}{(a^2 - x^2)^{\frac{3}{2}}} = \frac{1}{\sqrt{a^2 - x^2}} + C.$$

4.] To integrate

$$du = pnx^{n-1}dx\ (a^n + x^n)^{p-1},$$

we have immediately by form (7)

$$u = (a^n + x^n)^p + C.$$

5.] To integrate

$$(bx^2)^p \,.\, mx\ dx.$$

If we had $(bx^2)^p \,.\, 2bx\ dx$, we should have directly, by form (7), its integral

$$= \frac{1}{p+1} (bx^2)^{p+1} + C.$$

and therefore

$$\int (bx^2)^p \,.\, n \,.\, 2bx\ dx = \frac{n}{p+1} (bx^2)^{p+1} + C.$$

Let n be assumed $= \frac{m}{2b}$, or $n2b = m$, and we consequently have

$$\int (bx^2)^p mx\ dx = \frac{m}{2b\,(p+1)} (bx^2)^{p+1} + C.$$

6.] We have directly by the form (7)

$$\int (a + bx + cx^n)^p\ (bdx + ncx^{n-1}dx)$$

$$= \frac{1}{p+1} (a + bx + cx^n)^{p+1} + C.$$

7.] If the given differential be

$$du = (ax^n + b)^m x^{n-1} dx,$$

(m being a whole number), we may expand the binomial and integrate each term. But we may obtain the integral without going through this process; for if we write

$$ax^n + b = z,$$

we have $nax^{n-1}dx = dz$; whence

$$x^{n-1}dx = \frac{dz}{na}.$$

Also $du = z^m x^{n-1} dx = \frac{z^m dz}{na}$;

whence $u = \frac{z^{m+1}}{(m+1)\,na}$;

or restoring the value of z,

$$u = \frac{(ax^n + b)^{m+1}}{(m+1)na} + C.$$

If the differential be

$$du = \frac{cx^m dx}{(ax+b)^m},$$

writing $ax + b = z$, we have $x = \frac{z-b}{a}$,

and $dx = \frac{dz}{a}$.

Consequently $du = \frac{\frac{c\,(z-b)^m}{a^m} \cdot \frac{dz}{a}}{z^m} = \frac{c\,(z-b)^m \cdot dz}{a^{m+1} z^m}$.

Here we have only to expand the binomial and divide by z^m, whence we obtain a series of simple terms,

each of which is to be integrated separately, and the sum of the integrals taken.

8.] To integrate

$$du = (a + bx + cx^2)^{\frac{2}{3}}(bdx + 2cx\,dx).$$

This comes under form (7.) For if we write

$$a + bx + cx^2 = z \qquad \text{then} \quad bdx + 2cx\,dx = dz$$

and $du = z^{\frac{2}{3}}dz$. Here $n = \frac{2}{3}$ $n + 1 = \frac{2}{3} + 1 = \frac{5}{3}$:

whence the integral is, by form (7),

$$u = \frac{z^{\frac{5}{3}}}{\frac{5}{3}} + C = \frac{3}{5}z^{\frac{5}{3}} + C$$

$$= \frac{3}{5}(a + bx + cx^2)^{\frac{5}{3}} + C.$$

9.] To integrate

$$du = \sqrt{a + bx^2}\,mx\,dx,$$

writing $a + bx^2 = z$ $\qquad 2bx\,dx = dz,$

which differs from the other factor only in the constant; hence it comes under form (8), or we have, $n = 2$ $p = \frac{1}{2}$,

and $u = \dfrac{m}{2b} \cdot \dfrac{1}{\frac{1}{2}+1} \cdot z^{\frac{1}{2}+1} + C = \dfrac{m}{3b} \cdot z^{\frac{3}{2}} + C.$

$$= \frac{m}{3b}(a + bx^2)^{\frac{3}{2}} + C.$$

10.] To integrate $\frac{dx}{x^n}$ this may evidently be written,

$$=\frac{(n-1)dx}{(n-1)x^n}=\frac{1}{(n-1)}\cdot\frac{(n-1)x^{n-2}dx}{x^{2n-2}}.$$

But the second factor is readily seen to be the differential of $\frac{-1}{x^{n-1}}$.

Hence we have the result,

$$\int\frac{dx}{x^n}=\frac{-1}{(n-1)x^{n-1}}+C.$$

11.] To integrate $dx\left(\frac{x^{n-1}}{(1+x)^{n+1}}\right)$.

This may be written,

$$=\frac{dx}{n}\,\frac{nx^{n-1}+nx^n-nx^n}{(1+x)^{n+1}}.$$

$$=\frac{dx}{n}\,\frac{(1+x)nx^{n-1}-nx^n}{(1+x)^{n+1}},$$

or multiplying by $(1+x)^{n-1}$

$$=\frac{dx}{n}\,\frac{(1+x)^n nx^{n-1}-(1+x)^{n-1}nx^n}{(1+x)^{2n}},$$

which puts it into the form of a simple differential, and we have

$$\int dx\frac{x^{n-1}}{(1+x)^{n+1}}=\frac{1}{n}d\cdot\frac{x^n}{(1+x)^n}+C.$$

In general, in the subsequent examples, it is to be understood that a constant remains to be added to the integral if it be not expressed.

12.] Let it be required to find

$$u = \int dx \int dx \int dx \int \&c. \textit{ ad infinitum}.$$

Differentiation gives us

$$du = dx \int dx \int dx \int \&c.$$
$$= dx \,.\, u;$$

whence we have

$$\frac{du}{u} = dx,$$

which gives $x = \log. u$, or $u = \epsilon^x$.

13.] If it be required to find the same to n terms, we must proceed thus:

$$\frac{x^2}{2} = \int x dx = \int \left(\int dx \right) dx.$$

And since it is indifferent in what order the integrations are performed, this may be written

$$= \int dx \int dx.$$

In like manner

$$\frac{x^3}{2 \,.\, 3} = \int \frac{x^2}{2} dx = \int \left(\int x dx \right) dx,$$

which by substituting the last value in the same way,

$$= \int dx \int dx \int dx.$$

In the same manner

$$\frac{x^4}{2 \,.\, 3 \,.\, 4} = \int dx \int dx \int dx \int dx.$$

And proceeding thus we shall obtain

$$\frac{x^n}{1.2.3.4\ldots\ldots n}=\int dx\int dx\int dx\int dx, \text{ \&c. to } n \text{ terms.}$$

14.] We can express by a series the integral of $\dfrac{x^m dx}{a^n+x^n}$ by expanding the coefficient, which gives

$$\frac{x^m}{a^n+x^n}=\frac{x^m}{a^n}-\frac{x^{m+n}}{a^{2n}}+\frac{x^{m+2n}}{a^{3n}}-\text{\&c.}$$

whence we have

$$\int\frac{x^m dx}{a^n+x^n}=\frac{x^{m+1}}{a^n(m+1)}-\frac{x^{m+n+1}}{a^{2n}(m+n+1)}+\text{\&c.}$$

This series, however, will not converge unless x be small: and in all cases when we wish to integrate by series, this is a point of primary importance: it therefore becomes highly desirable to possess several forms of series for the same function, some of which may be suited to the case where x is small, and others to that where it is a large quantity: in this last case we ought to have a descending series, or one involving negative powers of x.

In the present instance we can readily obtain such a series, from the simple consideration of writing the binomial x^n+a^n, and developing the fraction in this form: which gives

$$\frac{x^m}{x^n+a^n}=\frac{x^m}{x^n}-\frac{x^m a^n}{x^{2n}}+\frac{x^m a^{2n}}{x^{3n}}-\text{\&c.}$$

$$=\frac{1}{x^{n-m}}-\frac{a^n}{x^{2n-m}}+\frac{a^{2n}}{x^{3n-m}}-\text{\&c.}$$

and integrating these terms by applying the formula,

$$\int \frac{dx}{x^n} = \frac{-1}{(n-1)x^{n-1}},$$

we have

$$\int \frac{x^m dx}{x^n + a^n} = -\frac{1}{(n-m-1)x^{n-m-1}} + \frac{a^n}{(2n-m-1)\,x^{2n-m-1}} - \&c.$$

another expression for the same integral which converges in proportion as x increases.

If in the developement previous to integration the index of x in any of the denominators become $=1$, or $rn-m=1$; that is, $rn=m+1$, or $(m+1)$ be a multiple of n, the integration of that term is reduced to the logarithmic integral, or $=a^{(r-1)n} \log. x$.

15.] We can integrate by a series the form

$$\frac{dx}{\sqrt{(1-x^2)(a+x)}}.$$

For the coefficient being written

$$\frac{1}{\sqrt{1-x^2} \cdot \sqrt{a+x}} = \frac{1}{\sqrt{1-x^2}} \cdot \sqrt{a+x} \cdot \frac{1}{a+x},$$

if we expand $\sqrt{a+x}$ by the binomial theorem, it will give a series of the form

$$A + Bx + Cx^2 + \&c.$$

Again, if we expand $\dfrac{1}{a+x}$ we shall have another series of the form

$$\alpha + \beta x + \gamma x^2 + \&c.,$$

and the product of these will evidently give another series of the same form. Thus upon the whole we shall have to integrate a series of terms

$$\frac{k}{\sqrt{1-x^2}}+\frac{mx}{\sqrt{1-x^2}}+\frac{nx^2}{\sqrt{1-x^2}}+\&c.$$

the integrals of which have been given in the examples of binomial differentials. [Calc. and Curves Introd.]

We can easily bring under this form the expression

$$\frac{du}{\sqrt{(2mu-u^2)(n-u)}},$$

(which occurs in some mechanical investigations,) by writing $(m-u)=mx$: whence $u=m\ (1-x)$,

and $\quad du=-mdx$;

also $\quad \frac{n-m}{m}=a$: whence $n=m\ (a+1.)$

The expression thus becomes

$$\frac{-mdx}{\sqrt{[2m^2(1-x)-m^2(1-x)^2]m(a+1)-m(1-x)}},$$

which is immediately reducible, by expanding $(1-x)^2$, to

$$\frac{-dx}{\sqrt{m(1-x^2)(u+x)}},$$

which is the form just integrated by a series.

RATIONAL FRACTIONS.

(1.) The preliminary operation of reducing the dimensions of the numerator lower than those of the denominator by division, (Int. Calc. p. 100.) will give us a series of simple terms, and in many cases also a final term, which can be integrated at once by elementary forms. Thus, to integrate

$$du = \frac{xdx}{a+bx}$$

proceeding to develope the quotient, we have only to take one term of the series before we arrive at a final term, which is an elementary form; for we have,

$$du = dx\left(\frac{x}{bx+a}\right)$$

$$= dx\left(\frac{1}{b} - \frac{a}{b}\,\frac{1}{bx+a},\right)$$

and observing that $\frac{dx}{bx+a} = \frac{1}{b} \,.\, d \,.\, \log. \,(bx+a)$, we find

$$u = \frac{x}{b} - \frac{a}{b^2} \log. \,(bx+a) + C.$$

In a similar way we may integrate

$$\frac{x^3dx}{a+bx^2} \text{ \&c.}$$

(2.) To take a more general form;—let it be required to integrate

$$du = \frac{x^p dx}{a + bx^n};$$

where $(p+1)$ is a multiple of n, suppose it $= rn$; whence $p = rn - 1$, and we may write the expression,

$$du = dx \frac{x^{rn-1}}{b} \left\{ \frac{1}{x^n + \frac{a}{b}} \right\}.$$

Here, expanding the second factor, we shall have to replace it by the series

$$\frac{1}{x^n} - \frac{a}{bx^{2n}} + \frac{a^2}{b^2 x^{3n}} - \&c. \ldots . \frac{\frac{a^{r-1}}{b^{r-1} x^{(r-1)n}}}{x^n + \frac{a}{b}},$$

or we have

$$du = dx \frac{1}{b} \left\{ x^{rn-n-1} - \frac{a}{b} x^{rn-2n-1} + \ldots \frac{a^{r-1}}{b^{r-1}} \frac{x^{rn-(r-1)n-1}}{x^n + \frac{a}{b}} \right\}$$

$$= dx \frac{1}{b} \left(x^{(r-1)n-1} - \frac{a}{b} x^{(r-2)n-1} + \ldots \frac{a^{r-1}}{b^{r-1}} \cdot \frac{bx^{n-1}}{bx^n + a} \right);$$

whence, by integrating each term, we find

$$u = \frac{1}{b} \left(\frac{x^{(r-1)n}}{(r-1)n} - \frac{a}{b} \frac{x^{(r-2)n}}{(r-2)n} + \ldots \frac{a^{r-1}}{b^{r-1}} \cdot \frac{1}{n} \log.(bx^n + a) \right).$$

(3.) Proceeding now to those cases where, after this preliminary operation, we have to apply the principle of resolving the denominator of the fractional coefficient into its factors, in order to illustrate more satisfactorily the general view given in the former part of the work, we will take a few simple examples of each of the cases there considered.

Let it be required to integrate

$$\frac{dx}{x^2 - a^2}.$$

Here the denominator is immediately resolvable into its simple factors, or the expression may be written

$$dx \cdot \frac{1}{(x+a)(x+a)}.$$

Thus we may resolve the coefficient into its "partial "fractions"

$$\frac{A}{x+a} + \frac{B}{x-a}.$$

These two fractions reduced to a common denominator must be equal to the former, or,

$$\frac{A(x-a) + B(x+a)}{(x+a)(x-a)} = \frac{1}{(x+a)(x-a)}.$$

This gives us, (in order to find the values of A and B)

$$A(x-a) + B(x+a) - 1 = 0,$$

or $$(A+B)x + (B-A)a - 1 = 0 \quad . \ . \ . \ . \ . \ . \ . \quad (\alpha)$$

But when $x = 0$, the former expression becomes $\frac{1}{-a^2}$: and on the same supposition the latter form becomes $\frac{(B-A)a}{-a^2}$, consequently we must have $(B-A)a = 1$, which reduces the equation (α) to $(A+B)x = 0$, whatever be the value of x; that is, we have $A + B = 0$, or $A = -B$.

Hence by substitution $2Ba = 1$,

or $$B = \frac{1}{2a}.$$

Thus upon the whole the expression will be

$$\left(\frac{B}{x-a}-\frac{B}{x+a}\right)dx=\frac{1}{2a}\left(\frac{1}{x-a}-\frac{1}{x+a}\right)dx.$$

Hence the integral is immediately found to be

$$\frac{1}{2a}\log.\left(\frac{x-a}{x+a}\right).$$

We have stated this process at length, in order to illustrate the theory: but both in this, and in many other cases, shorter methods may be found by the adoption of various artifices. Thus in the present case the given function may obviously be put under the forms

$$\frac{1}{2a}\left(\frac{2a}{x^2-a^2}\right)=\frac{1}{2a}\left(\frac{x+a-(x-a)}{x^2-a^2}\right)$$

$$=\frac{1}{2a}\left(\frac{1}{x-a}-\frac{1}{x+a}\right),$$

whence the integral is obtained as before.

(4.) Let it be required to integrate the function

$$\frac{axdx}{(x-a)^2}$$

which has two equal roots. Here, [Diff. Calc. p. 102.] writing $x-a=z$, we have $x=z+a$, and $dx=dz$; and the expression becomes

$$dz\,.\left(\frac{az+a^2}{z^2}\right).$$

Hence we have the partial fractions

$$dz\left(\frac{a}{z}+\frac{a^2}{z^2}\right),$$

and the integral will be

$$a \log. z - \frac{a^2}{z},$$

or restoring the value of z, it becomes

$$a \log. (x-a) - \frac{a^2}{x-a}.$$

(5.) To integrate

$$du = \frac{xdx}{1+2ax-x^2},$$

the factors of the denominator are easily found, by solving the quadratic $x^2 - 2ax = 1$,

which gives $x - a = \pm\sqrt{1+a^2}$,

or the two roots are $x = a + \sqrt{1+a^2}$,

and $x = a - \sqrt{1+a^2}$.

Hence the given differential becomes

$$dx\left\{\frac{x}{(x+a+\sqrt{1+a^2})(x+a-\sqrt{1+a^2})}\right\}$$

$$= dx\left\{\frac{A}{x+a+\sqrt{1+a^2}} + \frac{B}{x+a-\sqrt{1+a^2}}\right\}.$$

The numerator, after reducing to a common denominator, will be

$$A(x+a-\sqrt{1+a^2}) + B(x+a+\sqrt{1+a^2}) = x,$$

or $(A+B)x + (A+B)a + (B-A)\sqrt{1+a^2} = x.$

whence we deduce

$$(A+B) = 1, \text{ and } (A+B)a + (B-A)\sqrt{1+a^2} = 0.$$

whence $B=1-A$, and $a+(1-2A)\sqrt{1+a^2}=0$;

$$\text{thence } 2A=\frac{a+\sqrt{1+a^2}}{\sqrt{1+a^2}},$$

$$\text{and } A=\frac{a+\sqrt{1+a^2}}{2\sqrt{1+a^2}}.$$

Hence, by substituting these values, we find

$$B=1-A=\frac{2\sqrt{1+a^2}-a-\sqrt{1+a^2}}{2\sqrt{1+a^2}}$$

$$=\frac{\sqrt{1+a^2}-a}{2\sqrt{1+a^2}}.$$

But observing that $d\,(x+a\pm\sqrt{1+a^2})=dx$, we have the integral

$$u=A\log.(x+a+\sqrt{1+a^2})+B\log.(x+a-\sqrt{1+a^2}$$

In which we have only to substitute the values of the constants A and B, as found above.

(6.) We will now proceed to an example of a function whose denominator contains three roots, two of which are equal. Let the function be

$$dx\,.\,\frac{1}{x^3+3x^2-4},$$

whose denominator is easily seen to be formed by multiplying together the factors

$$(x+2)\,(x+2)\,(x-1).$$

Hence we have the partial fractions

$$\frac{A}{(x+2)}+\frac{B}{(x+2)}+\frac{C}{(x-1)};$$

and reducing to a common denominator, we find that the first two terms may be put under the form

$$\frac{Dx+E}{(x+2)^2},$$

and the whole numerator will be

$$Dx^2+Ex-Dx-E+Cx^2+4Cx+4C;$$

which being arranged and written $=1$, we have

$$(D+C)\,x^2+(E-D+4C)\,x+4C-E-1=0.$$

Hence $(D+C)=0 \therefore D=-C$

$$E-D+4C=E-5D=0 \therefore E=5D$$

$$E=4C-1 \therefore =5D=-5C$$

$$\therefore (4+5)\,C=1, \text{ or } C=\frac{1}{9} \therefore D=-\frac{1}{9}, \text{ and } E=-\frac{5}{9}.$$

With these values of the coefficients, we proceed to integrate; and taking the expression

$$\left\{\frac{Dx}{(x+2)^2}+\frac{E}{(x+2)^2}+\frac{C}{x-1}\right\}dx,$$

we may for $(x+2)$ substitute y, which gives $dy=dx$, and $x=y-2$; this will put the above expression, with the substitution of the values of the constants before obtained, into the form

$$dy\left\{-\frac{1}{9}\left[\frac{y}{y^2}-\frac{2}{y^2}\right]-\frac{5}{9}\cdot\frac{1}{y^2}\right\}+\frac{1}{9}\cdot\frac{dx}{x-1}.$$

The first member of which is

$$-\frac{1}{9}\cdot\frac{dy}{y}+\frac{2-5}{9}\cdot\frac{dy}{y^2}, \text{ or } -\frac{1}{9}\frac{dy}{y}-\frac{1}{3}\frac{dy}{y^2}.$$

And restoring the value of y, the whole integral will be

$$\frac{1}{9}\log.(x-1)-\frac{1}{9}\log.(x+2)+\frac{1}{3}\cdot\frac{1}{x+2},$$

$$\text{or } \frac{1}{3\,(x+2)}+\frac{1}{9}\log.\left[\frac{x-1}{x+2}\right].$$

(7.) For another example of a fraction whose denominator involves both equal and unequal roots, we may take

$$\frac{dx}{(x-a)^2\,(x-b)}.$$

Here, as before, writing $x-a=z$,

we have $x=z+a$,

and the coefficient becomes

$$\frac{1}{z^2\,(z+a-b)};$$

which separated into its partial fractions,

$$=\frac{A}{z^2}+\frac{B}{z+a-b} \quad \ldots\ldots\ldots\ldots \quad (\alpha)$$

and these being reduced to a common denominator,

$$=\frac{Az+Aa-Ab+Bz^2}{z^2\,(z+a-b)}$$

$$=\frac{Bz^2}{z^2\,(z+a-b)}+\frac{Az}{z^2\,(z+a-b)}+\frac{A\,(a-b)}{z^2\,(z+a-b)};$$

$$\text{or } =\frac{B}{z+a-b}+\frac{A}{z\,(z+a-b)}+\frac{A\,(a-b)}{z^2\,(z+a-b)}.$$

But comparing this with the original form, the last term alone without its coefficient is equal to the whole value: hence

$$A\,(a-b)=1, \text{ or } A=\frac{1}{a-b},$$

and the sum of the two first must $=0$;

or reducing them to the same denominator, the sum of their numerators $=0$,

$$\text{or } Bz+A=0 \;\therefore\; B=-\frac{A}{z}.$$

or substituting the value of A,

$$-B = -\frac{1}{z(a-b)}.$$

Hence the form (α) becomes

$$\frac{1}{a-b} \cdot \frac{1}{z^2} - \frac{1}{z(a-b)} \cdot \frac{1}{z+a-b}$$

$$= \frac{1}{a-b}\left[\frac{1}{z^2} - \frac{1}{z(z+a-b)}\right].$$

The first term is integrated directly, and the second, having two unequal factors, is found by the preceding process. For the integral of the first term we have obviously

$$\frac{-1}{z} = \frac{-1}{(x-a)},$$

and for the second term, or $\frac{1}{z(z+a-b)}$,

which is $= \frac{1}{(x-a)(x-b)}$,

we readily see that it is equal to

$$\frac{1}{a-b}\left\{\frac{(x-b)-(x-a)}{(x-a)(x-b)}\right\} = \frac{1}{a-b}\left\{\frac{1}{x-a} - \frac{1}{x-b}\right\};$$

the integral of which is $\frac{1}{a-b}\log.\left[\frac{x-a}{x-b}\right]$.

Thus upon the whole the entire integral will be

$$\frac{1}{a-b}\left\{\frac{-1}{x-a} - \frac{1}{a-b}\log.\left[\frac{x-a}{x-b}\right]\right\}.$$

(8.) For an example of a function containing ***imaginary roots***, let us take

$$du = \frac{(a+bx)\,dx}{x^3-1}.$$

The denominator is easily resolved into factors, one of the first, and the other of the second degree; or we have

$$x^3-1=(x-1)\ (x^2+x+1),$$

and the second factor is again resolvable into imaginary factors of the first degree, or

$$x^3-1=(x-1)\ (x-\tfrac{1}{2}+\sqrt{-\tfrac{3}{4}})\ (x-\tfrac{1}{2}-\sqrt{-\tfrac{3}{4}}.)$$

Proceeding according to the method laid down for expressions of this kind, (Int. Calc. p. 103,) we have

$$du=\frac{Adx}{x-1}+\frac{(Mx+N)\,dx}{x^2+x+1};$$

and on reducing to a common denominator, the numerator becomes

$$Ax^2+Ax+A+Mx^2-Mx+Nx-N=a+bx$$
$$=(A+M)\,x^2+(A-M+N)\,x+A-N.$$

Hence we obtain the values of the constants;

$$A+M=0 \qquad A-M+N=b \qquad A-N=a$$
$$M=-A \qquad 3A-a=b \qquad N=A-a$$
$$3A=b+a \quad A=\frac{b+a}{3} \qquad M=-\frac{b+a}{3}$$
$$N=\frac{b+a}{3}-a=\frac{b-2a}{3}.$$

Thus we shall have for the integral of the first member of the given expression

$$\int\frac{Adx}{x-1}=\frac{a+b}{3}\log.(x-1).$$

And for the second member we shall proceed thus:

The quantity (x^2+x+1), compared with the form (Int. Calc. p. 99) for imaginary factors, gives

$$2\alpha = 1 \qquad \alpha^2 + \beta^2 = 1$$

whence $\alpha = \frac{1}{2}$ $\beta^2 = 1 - \alpha^2 = 1 - \frac{1}{4}$ $\therefore$ $\beta = \sqrt{\frac{3}{4}}$.

Hence by form (33)

$$\int \frac{Mx+N}{x^2+x+1} = M.\log.\sqrt{x^2+x+1} + \frac{N - M\frac{1}{2}}{\sqrt{\frac{3}{4}}}\tan.^{-1}\frac{x+\frac{1}{2}}{\sqrt{\frac{3}{4}}}$$

$$= \left\{ \begin{array}{l} -\dfrac{b+a}{3}\log.\sqrt{x^2+x+1} \\ \qquad + \left(\dfrac{b-2a}{3} + \frac{1}{2}\dfrac{b+a}{3}\right)\dfrac{1}{\sqrt{\frac{3}{4}}}\tan.^{-1}\dfrac{x+\frac{1}{2}}{\sqrt{\frac{3}{4}}} \end{array} \right.$$

$$= -\frac{b+a}{3}\log.\sqrt{x^2+x+1} + \frac{b-a}{\sqrt{3}}\tan.^{-1}\left(\frac{2x+1}{\sqrt{3}}\right)$$

Adding this therefore to the integral of the first member, we have the entire integral of the given expression.

IRRATIONAL FUNCTIONS.

Many differentials affected by irrational coefficients fall under some of the elementary forms, or are directly reducible to them: of these we have already given some instances; we will here add the following:

(1.) The expression $\dfrac{adx}{(a^2+x^2)^{\frac{3}{2}}}$

may evidently be written in the form

$$adx \,.\, \frac{1}{a^2}\frac{a^2+x^2-x^2}{(a^2+x^2)^{\frac{5}{2}}}$$

$$= \frac{1}{a}dx\left\{\frac{\left(\sqrt{a^2+x^2} - \dfrac{x^2}{\sqrt{a^2+x^2}}\right)}{a^2+x^2}\right\}$$

This is easily seen to arise from differentiating

$$\frac{1}{a} \cdot \frac{x}{\sqrt{a^2+x^2}};$$

which is consequently the integral required.

(2.) To integrate

$$du = \frac{(1-z)\,dx}{\sqrt{1-z^2}},$$

we have only to write the expression in the form

$$du = \frac{dz}{\sqrt{1-z^2}} - \frac{zdz}{\sqrt{1-z^2}};$$

and we find immediately

$$u = -\cos.^{-1}z + \sqrt{1-z^2} + C.$$

This gives the integration of the form

$$dt = -a\sqrt{\frac{a}{\phi}}\left(\frac{(1-ez)\,dz}{\sqrt{1-z^2}}\right)$$

which occurs in dynamics.

(3.) The general principle of substituting another quantity which is a rational function of the variable, (Int. Calc. p. 112,) is easily applied in the following simple cases:

Let $du = \dfrac{dx}{\sqrt{x}\sqrt{1-x}}$

assuming $y = \sqrt{x}$, and therefore $x = y^2$,

and $dx = 2ydy$, the form becomes

$$\frac{2ydy}{y\sqrt{1-y^2}} = \frac{2dy}{\sqrt{1-y^2}};$$

and we have directly

$$u = 2 \sin.^{-1}y = 2 \sin.^{-1}\sqrt{x}.$$

(4.) To find $\int \frac{dv}{v^{\frac{1}{2}} + v^{\frac{3}{2}}} = \int \frac{dv}{v^{\frac{1}{2}}(1+v)}$

let $\quad v = x^2 - 1,$

whence $\quad dv = 2x\,dx,$

then the form becomes

$$\int \frac{2x\,dx}{\sqrt{x^2-1}\,.\,x^2} = 2\int \frac{dx}{x\sqrt{x^2-1}}$$

$$= 2\,.\,\sec.^{-1}x\,;$$

which, since $x^2 = 1 + v$, and therefore $x = \sqrt{1+v}$,

becomes $\quad = 2\,.\,\sec.^{-1}\sqrt{1+v}.$

(5.) To find $\int \frac{\sqrt{x}dx}{x-1}$.

Let $\sqrt{x} = z \therefore x = z^2,\ dx = 2zdz$

$$\therefore \frac{\sqrt{x}dx}{x-1} = \frac{2z^2dz}{z^2-1},$$

which may be written

$$dz\left(\frac{2z^2-2+2}{z^2-1}\right)$$

$$= dz\left(2 + \frac{2}{z^2-1}\right);$$

and the second term of this coefficient is

$$\frac{2}{(z+1)(z-1)} = \frac{(z+1)-(z-1)}{(z+1)(z-1)}$$

$$= \frac{1}{z-1} - \frac{1}{z+1};$$

hence the expression becomes

$$= dz\left(2 + \frac{1}{z-1} - \frac{1}{z+1}\right),$$

and $\int \frac{2z^2dz}{z^2-1} = 2z + \log.(z-1) - \log.(z+1)$,

or $\int \frac{\sqrt{x}dx}{x-1} = 2\sqrt{x} + \log.\left(\frac{\sqrt{x}-1}{\sqrt{x}+1}\right)$.

(6.) To integrate $\frac{dx}{x^2\sqrt{1-x^2}}$

we may write the expression

$$dx\left(\frac{x^2+(1-x^2)}{x^2\sqrt{1-x^2}}\right),$$

or dividing by $\sqrt{1-x^2}$

$$\frac{x \cdot \frac{xdx}{\sqrt{1-x^2}} + \sqrt{1-x^2}dx,}{x^2}$$

which is evidently the differential of

$$\frac{-\sqrt{1-x^2}}{x}.$$

(7.) To integrate the irrational function

$$\frac{dx}{x\sqrt{1+x^2}},$$

we have only to observe that the coefficient is obviously equivalent to

$$\frac{\sqrt{1+x^2}-1+x^2-x^2}{(\sqrt{1+x^2}-1)x\sqrt{1+x^2}},$$

whose numerator again is equal to

$$x^2-\sqrt{1+x^2}(\sqrt{1+x^2}-1);$$

so that the whole is the result of reducing to a common denominator the two fractions

$$\frac{x}{(\sqrt{1+x^2}-1)\sqrt{1+x^2}}-\frac{1}{x}.$$

And these, when multiplied by dx, become

$$\frac{\frac{xdx}{\sqrt{1+x^2}}}{\sqrt{1+x^2}-1}-\frac{dx}{x},$$

which are both integrated immediately by logarithms, and we have the whole integral

$$\log.(\sqrt{1+x^2}-1)-\log. x$$
$$=\log.\left(\frac{\sqrt{1+x^2}-1}{x}\right).$$

In like manner, if the term under the radical sign had been $1-x^2$, the result would have been the same, with only this substitution.

(8.) To integrate $du=\frac{dx}{x\sqrt{1-x}}$.

Let $1-x=y^2$, whence $-dx=2y\,dy$,

and $$du=\frac{-2y\,dy}{(1-y^2)y}=\frac{-2dy}{1-y^2},$$

which is integrated by the logarithmic form IV; by which we find

$$u=-\log.\left(\frac{1+y}{1-y}\right),$$

and restoring the value of y,

$$u = -\log.\left(\frac{1+\sqrt{1-x}}{1-\sqrt{1-x}}\right).$$

(9.) The irrational function

$$du = \frac{(1+\sqrt{x}-\sqrt[3]{x^2})}{1+\sqrt[3]{x}}dx,$$

comes under form (39,) (Int. Calc. p. 113,) and is easily rationalized by making $x = z^6$; this substitution will give $dx = 6z^5dz$, and the function will become

$$du = 6z^5dz\frac{(1+z^3-z^4)}{1+z^2}.$$

We have then only to expand the coefficient, which after multiplying by z^5 gives

$$6dz\left(-z^7+z^6+z^5-z^4+z^2+1-\frac{1}{1+z^2}\right)$$

and the integral will be

$$u = 6\left(-\frac{z^8}{8}+\frac{z^7}{7}+\frac{z^6}{6}-\frac{z^5}{5}+\frac{z^3}{3}+z-\tan.^{-1}z\right).$$

And replacing the value of $z = x^{\frac{1}{6}}$, it becomes

$$u = \left(-\frac{6}{8}x^{\frac{8}{6}}+\frac{6}{7}x^{\frac{7}{6}}+x-\frac{6}{5}x^{\frac{5}{6}}+\frac{6}{3}x^{\frac{3}{6}}+6x^{\frac{1}{6}}-\tan.^{-1}x^{\frac{1}{6}}\right).$$

(10.) The integration of the function

$$du = \frac{dx}{x\sqrt{a^2+bx}}$$

will come under the form (41) (Int. Calc. p. 114.) that is, if we substitute $y^2 = a^2+bx$,

Then $$2y\,dy = b\,dx, \text{ and } dx = \frac{2y\,dy}{b};$$

Also $$x = \frac{y^2 - a^2}{b},$$

Hence, substituting these values, we have

$$du = \frac{2y\,dy}{b} \cdot \frac{b}{y^2 - a^2} \cdot \frac{1}{y}$$

$$= \frac{2dy}{(y^2 - a^2)},$$

which is integrated directly by the logarithmic form IV.

(11.) Let it be required to integrate

$$du = \frac{dx}{\sqrt{a + bx - x^2}}.$$

We see that this quantity comes under the case of form V. of irrational fractions; (Int. Calc. p. 114.) and by the equations investigated in p. 118. we find

$$du = -\frac{2(\beta - \alpha)z \,.\, dz}{(z^2 + 1)^2 \,.\, z \,.\, \frac{\beta - \alpha}{z^2 + 1}}$$

$$= \frac{-2dz}{z^2 + 1}.$$

Whence we find

$$u = -2 \tan.^{-1} z + C,$$

or replacing the value of z, (p. 115.)

$$u = C - 2 \tan.^{-1} \left(\frac{\sqrt{(x - \alpha)(\beta - x)}}{x - \alpha} \right)$$

$$= C - 2 \tan.^{-1} \left[\frac{\sqrt{\beta - x}}{\sqrt{x - \alpha}}\right]$$

(12.) To find the integral of

$$du = dx\sqrt{2ax - x^2},$$

we compare this expression with that of the form, p. 117; where we find

$$\alpha = 0 \quad \beta = 2a,$$

and the equations, p. 118, become

$$\sqrt{x\,(2a - x)} = \frac{2az}{z^2 + 1}$$

$$dx = -\frac{4az}{(z^2 + 1)^2} \,.\, dz\,;$$

and these equations multiplied together give us

$$u = -\frac{8a^2z^2dz}{(z^2 + 1)^3}\,;$$

which being a rational fraction, is integrated by the method already delivered.

(13.) To integrate

$$Pdx \left(x + \sqrt{1 + x^2}\right)^{\frac{p}{q}},$$

where P is a function of x and of $\sqrt{1 + x^2}$,

the expression may be rationalized by assuming

$$x + \sqrt{1 + x^2} = u^q\,;$$

whence by squaring we obtain

$$x^2 + 2x\sqrt{1 + x^2} + 1 + x^2 = u^{2q}\,;$$

and since $\quad \sqrt{1 + x^2} = u^q - x,$

substituting this value, the last equation becomes

$$2x^2+2x\ (u^q-x)+1=u^{2q},$$

or $\quad 2u^q x = u^{2q}-1$

$$x=\frac{u^{2q}-1}{2u^q};$$

and differentiating, we obtain

$$dx=\frac{2u^q.\,2qu^{2q-1}du-(u^{2q}-1)\ 2qu^{q-1}du}{4u^{2q}},$$

which is easily reduced to

$$dx=\frac{(qu^{2q}+q)\ du}{2u^{q+1}};$$

also $\quad \sqrt{1+x^2}=u^q-x=\dfrac{u^{2q}+1}{2u^q}.$

Substituting these values in the original expression, it becomes

$$P\left(\frac{(qu^{2q}+q)\ du}{2u^{q+1}}\right)\ (u^q)^{\frac{p}{q}}$$

$$=Pdu\left(\frac{qu^{2q+p}+qu^p}{2u^{q+1}}\right)=P\,.\,du\,.\frac{q}{2}\left(u^{q+p-1}+u^{p-q-1}\right)$$

which is directly integrable.

BINOMIAL DIFFERENTIALS.

(14.) Binomial differentials are represented by the general form, [Int. Calc. (54.)]

$$x^{m-1}(a+bx^n)^{\frac{p}{q}}dx\,;$$

and we propose to investigate in what cases this expression can be made rational.

Let us assume $a+bx^n=z^q$,

then $(a+bx^n)^{\frac{p}{q}}=z^p$;

$$\text{hence } x^n=\frac{z^q-a}{b} \quad x=\left(\frac{z^q-a}{b}\right)^{\frac{1}{n}}$$

$$\therefore x^m=\left(\frac{z^q-a}{b}\right)^{\frac{m}{n}}$$

$$\text{also } dx=\frac{qz^{q-1}}{bnx^{n-1}}dz=\frac{qz^{q-1}}{bn\left(\frac{z^q-a}{b}\right)^{\frac{n-1}{n}}}dz\,;$$

$$\text{hence } x^{m-1}dx=\left(\frac{z^q-a}{b}\right)^{\frac{m-n}{n}}\frac{q}{bn}z^{q-1}dz.$$

Or upon the whole the form becomes

$$\frac{q}{bn}z^{p+q-1}\left(\frac{z^q-a}{b}\right)^{\frac{m}{n}-1}dz.$$

An expression which will evidently be rational whenever $\frac{m}{n}$ is a whole number, or when n is an aliquot part of m.

(15.) Another condition may also be obtained, under which it will appear that the expression can be rationalized. We may write it in this form;

$$x^{m-1}dx\ \left[(ax^{-n}+b)\ x^n\right]^{\frac{p}{q}}$$
$$=x^{m-1}\ dx\ (ax^{-n}+b)^{\frac{p}{q}}\ x^{\frac{np}{q}}$$
$$=x^{m+\frac{np}{q}-1}\ dx\ (ax^{-n}+b)^{\frac{p}{q}}.$$

And comparing this with the preceding form, the quantity $\left(m+\dfrac{np}{q}\right)$ in the index of x, corresponds to m in the former case, and the index of x within the brackets has still the same value: hence the condition of rationality becomes, that $\dfrac{m+\frac{np}{q}}{n}$ should be a whole number; or $\dfrac{m}{n}+\dfrac{p}{q}$, a whole number.

(16.) We will exemplify these principles in the following simple cases:

If we have given the binomial differential

$$x^8dx\ (a+bx^3)^{\frac{1}{3}},$$

comparing this with the formula, we find

$$m=9\ \ n=3\ \therefore\ \frac{m}{n}=3\ \ p=1\ \ q=3.$$

Hence it is transformed into

$$\frac{1}{b}\ z^3dz\ \left(\frac{z^3-a}{b}\right)^2,$$

$$\text{or}\ \ \frac{1}{b^3}\ \left[z^3\ (z^6-2az^3+a^2)\right]\ dz$$

$$=\frac{1}{b^3}\ (z^9-2az^6+a^2z^3)\ dz;$$

and the integral is

$$\frac{1}{b^3}\left(\frac{z^{10}}{10}-\frac{2az^7}{7}+\frac{a^2z^4}{4}\right).$$

In which, by substituting the value of z, we shall have the result in terms of x.

In like manner, by the second principle above laid down, the differential

$$x^4dx\ (a+bx^3)^{\frac{1}{3}},$$

is capable of being rationalized, since we have

$$\frac{m}{n}=\frac{5}{3}\ \frac{p}{q}=\frac{1}{3}\ \text{and}\ \therefore\ \frac{m}{n}+\frac{p}{q}=\frac{6}{3}=2.$$

(17.) There are numerous cases, however, to which the preceding method cannot be applied. In many of these we may proceed by one or the other, or both of the processes now to be explained, commonly called the "reduction of binomial differentials."

The given differential being

$$x^{m-1}\,dx\ (a+bx^n)^p,$$

when p represents any *fractional* index, it may also be written thus:

$$x^{m-n}\,x^{n-1}\,dx\ (a+bx^n)^p.$$

But the second factor, having the index of x without the brackets less by unity than that within, is integrated by the principle of simple powers: [Int. Calc. form (8.)] Hence comparing this form with

$$\int udv=uv-\int vdu,$$

and representing this integrable factor by dv, and x^{m-n} by u, we have

$$v=\frac{(a+bx^n)^{p+1}}{(p+1)nb}\qquad du=(m-n)\ x^{m-n-1}\,dx,$$

and the formula becomes by substitution

$$\int x^{m-1}\,dx\,(a+bx^n)^p$$

$$=\frac{x^{m-n}\,(a+bx^n)^{p+1}}{(p+1)\,nb}-\int\frac{(a+bx^n)^{p+1}}{(p+1)nb}(m-n)\,x^{m-n-1}\,dx.$$

But the term $\int x^{m-n-1}\,dx\,(a+bx^n)^{p+1}$

may be written $\int x^{m-n-1}\,dx\,(a+bx^n)^p.\,(a+bx^n)$

or performing the multiplication,

$$a\int x^{m-n-1}dx\,(a+bx^n)^p+b\int x^{m-1}dx\,(a+bx^n)^p\,.$$

Substituting this value in the foregoing equation, and collecting the terms

$$\left(1+\frac{m-n}{(p+1)n}\right)\int x^{m-1}dx\,(a+bx^n)^p$$

$$=\frac{x^{m-n}(a+bx^n)^{p+1}-a\,(m-n)\int x^{m-n-1}dx\,(a+bx^n)^p}{(p+1)\,nb}\,;$$

or, observing that the coefficient on the first side of this equation is equal to $\frac{pn+m}{(p+1)n}$, and dividing by this quantity, we have

$$\int x^{m-1}\,dx\,(a+bx^n)^p=$$

$$\frac{x^{m-n}\,(a+bx^n)^{p+1}-a\,(m-n)\int x^{m-n-1}dx(a+bx^n)^p}{b\,(pn+m)}\ldots(A).$$

(18.) By this formula then we reduce the determination of a function involving x^{m-1}, to one containing x^{m-n-1}, and these indices being arbitrary and general,

it is evident that we may thus continue the process till we come to an elementary form. In this way, if instead of m in the first index we had $m-n$, the second would be $m-2n-1$: and hence again, changing $m-n$ into $m-2n$, we should have for the second $m-3n-1$, and so on. If r denote the number of such successive reductions, we shall come at last to the index $m-rn-1$, or the last form of the expression will be

$$\int x^{m-(r-1)n-1}dx(a+bx^n)^p =$$

$$\frac{x^{m-rn}(a+bx^n)^{p+1}-a(m-rn)\int x^{m-rn-1}dx(a+bx^n)^p}{b(pn+m-(r-1)n)}.$$

Here if m be a multiple of n, we shall have some value of r, at which $m=rn$; in which case the formula becomes,

$$\int x^{n-1}dx(a+bx^n)^p = \frac{(a+bx^n)^{p+1}}{b(pn+n)},$$

or is reduced to the elementary expression for an algebraical power. [Int. Calc. form (8.)]

(19.) Another method by which binomial differentials may be successively reduced to lower indices, is as follows: we may write the expression

$$\int x^{m-1}dx(a+bx^n)^p.$$

In the form $\int x^{m-1}dx(a+bx^n)^{p-1}(a+bx^n)$,

which performing the multiplication puts it into the form

$$a\int x^{m-1}dx(a+bx^n)^{p-1} + b\int x^{m+n-1}dx(a+bx^n)^{p-1}.$$

But in the formula (A), if we change m into $m+n$, and p into $p-1$, we have the value of this last term,

$$\int x^{m+n-1}\,dx\,(a+bx^n)^{p-1} =$$

$$\frac{x^m(a+bx^n)^p - am\int x^{m-1}\,dx\,(a+bx^n)^{p-1}}{b\,(pn+m)}$$

Hence, by substituting this value, the preceding formula becomes

$$\int x^{m-1}\,dx\,(a+bx^n)^p$$

$$=\left\{\begin{array}{l} a\int x^{m-1}\,dx\,(a+bx^n)^{p-1} \\ +\dfrac{x^m\,(a+bx^n)^p - am\int x^{m-1}\,dx\,(a+bx^n)^{p-1}}{pn+m}\end{array}\right\}$$

and reducing to a common denominator, and adding these two terms, there results the formula,

$$\int x^{m-1}\,dx\,(a+bx^n)^p =$$

$$\frac{x^m(a+bx^n)^p + pna\int x^{m-1}\,dx\,(a+bx^n)^{p-1}}{pn+m} \quad \ldots (B).$$

By this formula we may successively diminish the index p by unity, that of x, however, remaining unaltered. By applying therefore one or both these methods to a given binomial differential, we may in a considerable number of cases reduce it to a form in which it may be rationalized or reduced to some of the elementary integrations.

(20.) It is evident that if m and p were negative, the formulæ (A) and (B) would not answer the purpose of diminishing the index of the part without the

brackets in the one instance, and of that within in the other. They are, however, easily put into a form which will meet this case.

By transposing the terms in formula (A) we have

$$\int x^{m-n-1}\,dx\,(a+bx^n)^p$$

$$=\frac{x^{m-n}(a+bx^n)^{p+1}-b\,(m+np)\int x^{m-1}dx\,(a+bx^n)^p}{a\,(m-n}.$$

If we change m into $(m+n)$, this becomes

$$\int x^{m-1}dx(a+bx^n)^p=$$

$$\frac{x^m(a+bx^n)^{p+1}-b\,(m+n+np)\int x^{m+n-1}dx(a+bx^n)^p}{am}\ldots\ldots(C).$$

In this form, if we have $-m$, the index $m+n-1$ becomes $-m+n-1$, or is diminished.

In a similar way, by reversing the formula (B), we shall obtain a corresponding formula: from (B) we have,

$$\int x^{m-1}\,dx\,(a+bx^n)^{p-1}$$

$$=\frac{-(x^m(a+bx^n)^p-(m+np)\int x^{m-1}\,dx\,(a+bx^n)^p)}{pna}$$

which, writing $p+1$ instead of p, becomes

$$\frac{-(x^m\,(a+bx^n)^{p+1}-(m+n+np)\int x^{m-1}dx(a+bx^n)^{p+1})}{(p+1)\,na}\ldots(D).$$

In which $p+1$ becomes $-p+1$, or is diminished when p is negative.

None of these formula can be applied when the denominators vanish; as in formula (A) when $m=-np$.

But in all such cases the function proposed is integrable by elementary forms.

(21.) Particular cases of these methods for the integration of binomial differentials have been given in another place. (Introd. to Calc. and Curves, p. 8.) We will, however, here add, by way of further exemplification, one class of such integrations not there noticed; namely, those coming under the formula (C), or where the index of m is negative.

Let it be required to find $\int \frac{dx}{x^m\sqrt{1-x^2}}$;

here the index $-m-1$ in the formula is to be changed into $-m$, and we have the integral

$$= \frac{-\sqrt{1-x^2}}{(m-1)\,x^{m-1}} + \frac{m-2}{m-1}\int \frac{dx}{x^{m-2}\sqrt{1-x^2}}.$$

We may here obtain successively the integrals corresponding to successive values of m: but it is evident that this reduction from one value to another can only be continued so long as we take $m > 1$, for if $m = 1$ the denominator vanishes.

But in this case we may find the integral by other means. The expression becomes

$$dx\frac{1}{x\sqrt{1-x^2}},$$

which is immediately recognized as the differential before obtained for

$$-\log.\left(\frac{1+\sqrt{1-x^2}}{x}\right).$$

If we call this integral (A), we may proceed from it to the other values of the expression; making m successively $=3$, $=5$, &c.; the formula gives,

$$\int \frac{dx}{x^3\sqrt{1-x^2}} = \frac{-\sqrt{1-x^2}}{2x^2} + \frac{1}{2}(A) = (B)$$

$$\int \frac{dx}{x^5\sqrt{1-x^2}} = \frac{-\sqrt{1-x^2}}{4x^4} + \frac{3}{4}(B) = (C)$$

&c. = &c.

A series of even powers will in like manner be obtained if we set out from $m=2$. In which case we have an elementary integral, whose value was found before, or,

$$\int \frac{dx}{x^2\sqrt{1-x^2}} = -\frac{\sqrt{1-x^2}}{x} \qquad = (\alpha).$$

And thence successively

$$\int \frac{dx}{x^4\sqrt{1-x^2}} = -\frac{\sqrt{1-x^2}}{3x^3} + \frac{2}{3}(\alpha) = (\beta)$$

$$\int \frac{dx}{x^6\sqrt{1-x^2}} = -\frac{\sqrt{1-x^2}}{5x^5} + \frac{4}{5}(\beta) = (\gamma)$$

&c = &c.

The student will find an excellent collection of examples in Garnier's Leçons de Calcul. Integral.

INTEGRATION OF

TRANSCENDENTAL FUNCTIONS.

The integration of differential expressions involving transcendental functions of the variable was not entered upon in the former part of this work. We shall here give a short view of the principles of such integrations, and illustrate the general methods by a few examples.

EXPONENTIAL FUNCTIONS.

(1.) The principles of differentiation give us

$$da^x = Aa^x dx$$

$$\text{or } da^x = a^x dx \log. a,$$

and therefore reciprocally

$$\int a^x \, dx = \frac{a^x}{\log. a},$$

This is the elementary integral of logarithmic functions; and by the combination of this with the foregoing methods we are enabled to integrate the general expression $a^x X dx$, in which X is a function of x. For this purpose, writing the expression thus, $X \,.\, a^x dx$,

and integrating by the method of parts, we shall have

$$\int X \,.\, a^x dx = \frac{X \,.\, a^x}{\log. a} - \int \frac{a^x}{\log. a}\, dX,$$

or writing

$$dX = X_2 dx,\ dX_2 = X_3 dx,\ \&c.$$

we have also

$$\int \frac{a^x}{\log. a}\, dX = \int \frac{X_2}{\log. a}\, a^x dx = \frac{X_2}{(\log. a)^2}\, a^x - \int \frac{a^x}{(\log. a)^2} dX_2;$$

whence, substituting this value in the place of the last term in the equation above, we shall obtain

$$\int X a^x dx = \frac{X \,.\, a^x}{\log. a} - \frac{X_2 a^x}{(\log. a)^2} + \int \frac{a^x}{(\log. a)^2}\, dX_2.$$

This operation being thus continued, we shall arrive at the developement

$$\int X a^x dx = a^x \left(\frac{X}{\log. a} - \frac{X_2}{(\log. a)^2} + \frac{X_3}{(\log. a)^3} - \frac{X_4}{(\log. a)^4} \right.$$
$$\left. \ldots\ldots \pm \frac{X_n}{(\log. a)^n} \mp \int \frac{a^x dX_n}{(\log. a)^n} \right):$$

and if, taking the series of the differential coefficients, X_2, X_3, $X_4 \ldots\ldots X_n$, the last of these coefficients be constant, we shall have $dX_n = 0$, and therefore the part under the integral sign will vanish, and we shall have a finite value.

(2.) We may arrive also at another developement of $\int a^x X dx$ in the following manner:

Making $\int X dx = P$, $\int P dx = Q$, $\int Q dx = R$, &c.

and integrating by the method of parts, we shall have

$$\int a^x X dx = a^x P - \int a^x \log. a \; P dx$$

$$\int a^x \log. a \, . \, P dx = a^x \log. a \, . \, Q - \int a^x (\log. a)^2 \, Q dx \, ;$$

substituting in the previous equation, it will become

$$\int a^x . X dx = a^x P - a^x \log. a \, . \, Q + \int a^x (\log. a)^2 Q dx \, ;$$

and continuing to integrate by parts, we shall have generally

$$\int a^x X dx = a^x \left[P - Q \log. a + R (\log. a)^2 - \&c.\right]$$
$$\pm \int Z a^x (\log. a)^n dx.$$

(3.) If the functions represented by P, Q, R, &c. involve x in any negative powers, the last term will always contain the integral of $\frac{a^x dx}{x}$; a transcendental function, the value of which we cannot determine in a finite form; for the exponents of x in the functions P, Q, R, &c. being successively diminished by unity, the last of these functions must be of the form $\frac{A}{x}$, and consequently the last integral will be

$$\int \frac{A a^x}{x} dx = A \int \frac{a^x dx}{x},$$

since A is constant.

We can, however, obtain an approximate value by substituting in the expression the developement of a^x, which, as we have seen, is,

$$1 + x \log. a + \frac{x^2}{2} (\log. a)^2 + \frac{x^3}{2 \, . \, 3} (\log. a)^3 + \&c.,$$

and then integrating each term separately.

If in the equation $\frac{du}{u} = d \log. u$, or $du = ud \log. u$, we make $u = x^y$, we shall have

$$dx^y = x^y d \log. x^y;$$

thus, whenever we can decompose a differential into two parts, one of which may be represented by x^y, and the other by $d \log. x^y$, the integral will be $x^y + C$.

(4.) The integration by parts may be applied also to the expression $Xdx\ (\log. x)^n$; for if we represent the integral of Xdx by X_2, we shall have

$$\int Xdx\ (\log. x)^n = X_2\ (\log. x)^n - n \int \frac{X_2}{x} dx\ (\log. x)^{n-1};$$

and this last integral may be made in its turn to depend on another of the form

$$X_3 dx\ (\log. x)^{n-2};$$

and so on successively.

(5.) For a first example we will take

$$X = x^m;$$

then the general method gives us,

$$\int x^m dx \log. x = \frac{x^{m+1}}{m+1} \log. x - \int \frac{x^{m+1}}{m+1} \cdot \frac{dx}{x} (\log. x)^0$$

$$= \frac{x^{m+1}}{m+1} \left(\log. x - \frac{1}{m+1} \right) = (A)$$

$$\int x^m dx\ (\log. x)^2 = \frac{x^{m+1}}{m+1} (\log. x)^2 - 2 \int \frac{x^{m+1}}{m+1} \frac{dx}{x} \log. x$$

$$= \frac{x^{m+1}}{m+1} (\log. x)^2 - \frac{2}{m+1} (A) = (B)$$

$$\int x^m dx\ (\log. x)^3 = \frac{x^{m+1}}{m+1} (\log. x)^3 - \frac{3}{m+1} (B)$$

&c. = &c.

(6.) Next, let it be required to integrate

$$\frac{dx}{x \log. x};$$

this function is evidently equal to $d \,.\, \log. x \,.\, \frac{1}{\log. x}$,

and this again is $\quad d \,.\, \log. [\log. x]$,

or the integral is $\quad \log. \log. x$,

which is often written $\log.^2 x$.

(7.) In the same way, the expression

$$\frac{dx}{x (\log. x)^2}$$

is seen to be equal to

$$\frac{d \log. x}{(\log. x)^2} = \frac{0 + d \log. x}{(\log. x)^2};$$

in which form it is immediately evident that it is the differential of the fraction $\frac{-1}{\log. x}$.

(8.) Let it be required to integrate

$$\frac{dx}{\log. x};$$

writing $u = \log. x$, we have

$$x = e^u = 1 + u + \frac{u^2}{2} + \frac{u^3}{2 \,.\, 3} + \&c.$$

$$\therefore\; dx = du + udu + \tfrac{1}{2} u^2 du + \&c.$$

$$\text{Hence} \int \frac{dx}{\log. x} = \int \left(\frac{du}{u} + \frac{udu}{u} + \frac{1}{2} \frac{u^2 du}{u} + \&c. \right)$$

$$= \log. u + u + \frac{u^2}{2 \,.\, 2} + \frac{u^3}{2 \,.\, 3 \,.\, 3} + \&c.$$

$$= \log. \log. x + \log. x + \frac{1}{2} \frac{(\log. x)^2}{1 \,.\, 2} + \frac{1}{3} \frac{(\log. x)^3}{1 \,.\, 2 \,.\, 3} + \&c.$$

(9.) To integrate

$$du = \frac{dx}{\sqrt{1-\epsilon^x}},$$

let $\epsilon^x = z$, thence $\epsilon^x dx = dz$, and $dx = \frac{dz}{z}$, and the expression becomes

$$du = \frac{dz}{z\sqrt{1-z}},$$

a form which has been before integrated. [Irrational Functions, 8.]

This gives the integration of the expression

$$dt = \frac{dk}{\sqrt{2n\phi}\sqrt{1-\epsilon^{-rk}}},$$

where n, ϕ, and r are constant: an expression which occurs in some mechanical investigations.

TRIGONOMETRICAL FUNCTIONS.

(1.) We before pointed out the remarkable connexion between circular and logarithmic functions by means of differentiation. The same result may be obtained by integration; and it is in this point of view that many writers introduce the notice of it. If it were required to integrate the expression

$$du = \frac{dx}{\sqrt{1-x^2}},$$

we have, by multiplying numerator and denominator by the factor $(x\sqrt{-1}+\sqrt{1-x^2})$

$$du = \left\{\frac{\frac{xdx\sqrt{-1}}{\sqrt{1-x^2}}+dx}{x\sqrt{-1}+\sqrt{1-x^2}}\right\}$$

and multiplying again by $\sqrt{-1}$

$$= \left\{\frac{\frac{-xdx}{\sqrt{1-x^2}}+\sqrt{-1}\,dx}{\sqrt{-1}\,(x\sqrt{-1}+\sqrt{1-x^2})}\right\}.$$

Where it is easily seen that the numerator is the differential of the denominator, and we consequently have

$$u = \frac{1}{\sqrt{-1}}\log.\ (x\sqrt{-1}+\sqrt{1-x^2}).$$

But the original differential is that of a circular arc u, in terms of its sine x, and consequently $\sqrt{1-x^2}$ is its cosine: or we have

$$u\sqrt{-1} = \log.\ (\sqrt{-1}\sin.\ u+\cos.\ u).$$

Hence we might deduce the same results as before.

Thus we might proceed to the deduction of the developements of $\sin.^n x$, $\cos.^n x$, in terms of the sines and cosines of the arc x and its multiples, as we did before (p. 50). These series afford the means of integrating differentials affected by coefficients involving the powers of the sine or cosine of the variable; as we shall now proceed to explain.

(2.) To integrate the expression $\cos.^m x dx$, in which m is a whole number, we must put for $\cos.^m x$ its deve-

lopement, in terms of the cosines of the multiple arcs, so that the whole will be reduced to the knowing how to integrate the expression cos. $mxdx$.

For this purpose, we must observe, that if, in the equation

$$d \sin. z = \cos. z dz,$$

we make $z = mx$, we shall have

$$d \sin. mx = \cos. mx \,.\, mdx;$$

and therefore

$$\int \cos. mxdx = \frac{\sin. mx}{m};$$

and, similarly, we should find that

$$\int \sin. mxdx = -\frac{\cos. mx}{m}.$$

(3.) If we wished to integrate sin. $^m xdx$, we might proceed in a similar way; or, otherwise, representing the complement of x by z, we should have

$$x = \tfrac{1}{2}\pi - z, \quad dx = -dz, \quad \sin. x = \cos. z,$$

which would therefore change sin. $^m xdx$ into $-\cos.^m zdz$, and we might integrate as above.

(4.) Taking the most general case $\sin.^m x \cos.^n xdx$; if m be even, we will put $m = 2p$, when we shall have to integrate

$$\sin.^{2p} x \cos.^n xdx = (1 - \cos.^2 x)^p \cos.^n xdx; \text{ and}$$

developing $(1 - \cos.^2 x)^p$, and multiplying by $\cos.^n xdx$, we shall obtain a series of terms, each of the form $\cos.^k xdx$, which we shall integrate as above.

If m be odd, we must put $m = 2p + 1$, when we shall have

$$\sin.^m x \cos.^n x dx = \sin.^{2p} x \cos.^n x \sin. x dx$$

$$= (1-\cos.^2 x)^p \cos.^n x(-d\cos. x)$$

making $\cos. x = z$, we shall change this expression into

$$-(1-z^2)^p z^n dz;$$

and p and n being, by hypothesis, whole numbers, we may develope the binomial and integrate.

(5.) Applying this process to the expressions

$$\frac{\cos.^m x dx}{\sin.^n x}, \quad \frac{\sin.^n x dx}{\cos.^m x},$$

since the second comes under the form of the other, by making $x = \frac{\pi}{2} - z$, we shall consider only the first: and if m be even, we may assume $m = 2p$, whence we shall have

$$\frac{\cos.^m x dx}{\sin.^n x} = \frac{(1-\sin.^2 x)^p dx}{\sin.^n x}$$

$$= \frac{1 - p\sin.^2 x + \left(p.\frac{p-1}{2}\right)\sin.^4 x^2 + \&c.}{\sin.^n x} dx,$$

an expression, the integral of which will depend on those of the forms $\sin.^q x dx$ and $\frac{dx}{\sin.^k x}$.

If m be odd, making $m = 2p+1$, we shall have

$$\frac{\cos.^m x dx}{\sin.^n x} = \frac{(1-\sin.^2 x)^p \cos. x dx}{\sin.^n x} = (1-p.\sin.^2 x + \&c.)\frac{\cos. x dx}{\sin.^n x},$$

an expression, the integral of which will depend on those of $\sin.^q x \cos. x dx$ and $\frac{dx \cos. x}{\sin.^k x}$.

The integrals of $\sin.^q x dx$ and $\sin.^q x \cos. x dx$ have

already been treated of; to integrate $\frac{dx \cos. x}{\sin.^k x}$ we must put $\sin. x = z$, whence $dx \cos. x = dz$, and consequently

$$\int \frac{dx \cos. x}{\sin.^k x} = \int \frac{dz}{z^k} = \int z^{-k} dz = \frac{z^{-k+1}}{1-k} + C.$$

With respect to the integral of $\frac{dx}{\sin.^k x}$, the same transformation will change this expression into $\frac{dz}{z^k(1-z^2)^{\frac{1}{2}}}$, a formula which is integrated by methods already given.

(6.) If, lastly, we have to integrate $\frac{dx}{\cos.^m x \sin.^n x}$ we must multiply the expression by $\cos.^2 x + \sin.^2 x$, a quantity equivalent to unity, when we shall have

$$\frac{dx}{\cos.^m x \sin.^n x} = \frac{dx}{\cos.^{m-2} x \sin.^n x} + \frac{dx}{\cos.^m x \sin.^{n-2} x},$$

by which the sum of the indices of the denominator will be diminished; and repeating the operation, and setting apart successively all the fractions, which in their denominators contain powers of the sine alone, or the cosine alone, (since we know how to integrate these fractions from what has preceded,) at the last operation we shall meet with terms still containing powers of the sine and cosine, or which will be of the following forms:

$$\frac{dx}{\cos. x \sin. x}, \quad \frac{dx}{\cos. x}, \quad \frac{dx}{\sin. x}.$$

(7.) To integrate $\frac{dx}{\cos. x \sin. x}$, we must multiply the

numerator by $\cos.^2 x + \sin.^2 x$, and we shall have

$$\frac{dx}{\cos. x \sin. x} = dx \cdot \frac{\cos. x}{\sin. x} + dx \frac{\sin. x}{\cos. x} = \frac{d.\sin. x}{\sin. x} - \frac{d.\cos. x}{\cos. x},$$

the integral of which is

$$\log. \sin. x - \log. \cos. x + \log. C = \log. C \tan. x.$$

(8.) To integrate $\frac{dx}{\sin. x}$, we must put $\cos. x = z$, and we shall have

$$dx = -\frac{dz}{\sin. x}, \quad \frac{dx}{\sin. x} = -\frac{dz}{\sin.^2 x} = -\frac{dz}{1 - z^2},$$

an expression integrable by the method of rational fractions.

(9.) In regard to $\frac{dx}{\cos. x}$, we shall suppose $\sin. x = z$, and we shall find

$$\int \frac{dx}{\cos. x} = \int \frac{dz}{\sqrt{1 - z^2}}.$$

(10.) In general, we may always transform expressions containing sines and cosines into others which do not contain them, by equating $\sin. x$ or $\cos. x$ to a new variable z.

For example, if in the expression $\sin.^m x \cos.^n x dx$, we suppose $\sin. x = z$, we shall have

$$\cos. x = \sqrt{1 - z^2}, \quad dx = \frac{dz}{\sqrt{1 - z^2}},$$

and substituting, we shall find

$$\sin.^m x \cos.^n x dx = z^m (1 - z^2)^{\frac{n}{2}} (1 - z^2)^{-\frac{1}{2}} dz = z^m (1 - z^2)^{\frac{n-1}{2}} dz,$$

an expression which comes under the form of binomial differentials.

(11.) The method of integration by parts may also be applied immediately to the expression $\sin.^m x \cos.^n x dx$, observing that in order to compare the expression with udv, we must decompose it thus:

$$\sin.^{m-1} x \cos.^n x \sin. x dx = -\sin.^{m-1} x d . \frac{\cos.^{n+1} x}{n+1}.$$

(12.) Lastly, trigonometrical formulæ may in some cases be employed with advantage. To integrate, for example, $\sin. mx \cos. nx dx$; since by trigonometry,

$$\sin. a . \cos. b = \tfrac{1}{2} \sin. (a+b) + \tfrac{1}{2} \sin. (a-b),$$

on comparing the expressions $\sin. mx \cos. nx dx$ with this formula, we find

$$\sin. mx \cos. nx dx = \tfrac{1}{2} \sin. [(m+n)x] dx + \tfrac{1}{2} \sin. [(m-n)x] dx,$$

and the integral will be as above,

$$C - \tfrac{1}{2} \frac{\cos. [(m+n)x]}{m+n} - \tfrac{1}{2} . \frac{\cos. [(m-n)x]}{m-n}.$$

These are some of the most fundamental methods for the integration of trigonometrical functions. But for a more complete view of the subject the student is referred to larger treatises.

In all cases of transcendental functions we may obtain an approximate integration by series.

(13.) We will now illustrate these methods by one or two examples.

It was observed above (5) that the integration of $\frac{dx}{\sin.^k x}$ was reducible to that of the binomial differentials of the general form $\frac{dz}{z^k \sqrt{1-z^2}}$, referring them to the integrals of those forms (Bin. Diff. 21.) we shall find the integrals of the series of trigonometrical forms

$$\int \frac{dx}{\sin. x} = \int \frac{dz}{z\sqrt{1-z^2}} = -\log. \left(\frac{1+\cos. x}{\sin. x}\right),$$

which by trigonometry $= \log. \tan. \frac{1}{2} x$

$$\int \frac{dx}{\sin.^2 x} = \int \frac{dz}{z^2\sqrt{1-z^2}} = \frac{-\cos. x}{\sin. x}$$

$$\int \frac{dx}{\sin.^3 x} = \int \frac{dz}{z^3\sqrt{1-z^2}} = \frac{-\cos. x}{2 \sin.^2 x} + \frac{1}{2} \log. \tan. \tfrac{1}{2} x.$$

&c. = &c.

Again, if in any of these expressions we consider z to represent cos. x and change the sign, we shall have the *integral of the corresponding functions involving cos.* x *and its powers.*

(14.) To integrate $du = \dfrac{dx}{\sin.^3 x \cos.^2 x}$

we have by the general form (6)

$$\int \frac{dx}{\sin.^3 x \cos.^2 x} = \int \frac{dx}{\sin.^3 x} + \int \frac{dx}{\sin. x \cos.^2 x}.$$

And again the last term

$$\int \frac{dx}{\sin. x \cos.^2 x} = \int \frac{dx}{\sin. x} + \int \frac{dx \sin. x}{\cos.^2 x}.$$

And here again the last term

$$\int \frac{dx \sin. x}{\cos.^2 x} = \int \frac{-d \cos. x}{\cos.^2 x} = -\frac{1}{\cos. x}.$$

We have then only to substitute the values of

$$\int \frac{dx}{\sin.^3 x} = \frac{-\cos. x}{2 \sin.^2 x} = \frac{1}{2}\int \frac{dx}{\sin. x}.$$

And $\int \frac{dx}{\sin. x} = \log. \tan. \frac{1}{2} x.$

Then upon the whole we have

$$u = -\frac{\cos. x}{2 \sin. ^2 x} + \frac{3}{2} \log. \tan. \frac{1}{2} x - \frac{1}{\cos. x} + C.$$

. (15.) By means of these results we can integrate

$$\epsilon^{ms} \, ds = 2h \cos. ^2\theta \, \frac{d\alpha}{\cos.^3 \alpha},$$

an expression which occurs in mechanics, in which s and α are the variables, m the reciprocal of the modulus of the common logarithms. Hence for the first member of the equation we have

$$\int \epsilon^{ms} \, ds = \frac{1}{m} \epsilon^{ms}.$$

And for the second, by the preceding formulæ

$$\int \frac{d\alpha}{\cos. ^3 \alpha} = \frac{\sin. \alpha}{2 \cos. ^2 \alpha} + \frac{1}{2} \log. \tan. \left(\frac{\pi}{4} + \frac{\alpha}{2}\right) + C.$$

INTEGRATION OF DIFFERENTIAL EQUATIONS.

In the very brief notice of Differential Equations given in the former part of this work, we shewed how such equations could be integrated in the following cases.

1st. When the variables can be immediately separated by common algebraic processes, and the integration reduced to

$$\int X dx + \int Y dy + C = 0.$$

2dly. Such forms as admit of having the variables separated so as to give integrations of the form

$$\int \frac{dx}{X} + \int \frac{dy}{Y} + C = 0.$$

Or of the form

$$\int X dx - \int \frac{dy}{Y} + C = 0.$$

3dly. Equations of the form

$$M dx + N dy,$$

where M and N are homogeneous functions of x and y.

4thly. Equations of the form

$$du = Mdx + Ndy,$$

which is a complete differential of two variables. The condition which shews that an equation can be thus integrated being that we have

$$\frac{dM}{dy} = \frac{dN}{dx}.$$

We purpose here to carry on the subject to such extent as may suffice to convey a general acquaintance with its nature, without entering upon investigations unsuitable to an elementary treatise.

We shall first examine a class of equations, which though not given in the form of complete differentials, are easily rendered such. This will readily be understood from the following instance:

Completion of differentials by a common factor.

(1.) If we take the differential equation

$$ydx - xdy = 0,$$

and compare it with the last general form of those just given, we find

$$M = y, \quad N = -x;$$

and hence proceeding to apply the criterion of integrability, we have

$$\frac{dM}{dy} = 1, \qquad \frac{dN}{dx} = -1.$$

The equation therefore is not a complete differential. But we may observe that it would become so if multiplied by $\frac{1}{y^2}$; for in this case we have

$$M=\frac{1}{y} \quad N=-\frac{x}{y^2},$$

and $$\frac{dM}{dy}=\frac{-1}{y^2}=\frac{dN}{dx};$$

or the condition of the criterion is fulfilled.

We shall find that there are many equations which may be thus rendered integrable by supplying a factor; and the object of our inquiry will be to determine the principles upon which such a factor can be discovered.

(2.) Let $Pdx+Qdy=0$ be the equation which is a complete differential, and z the factor by which it is multiplied to make it so; and which we will suppose a function both of x and y: thus both P and Q are multiples of z, or

$$P=Mz, \; Q=Nz;$$

and if we substitute these values in the preceding equation, and divide by z, we shall have

$$Mdx+Ndy=0 \quad \ldots\ldots\ldots\ldots\ldots \quad (a).$$

Now the equation $Pdx+Qdy=0$ being, by hypothesis, a complete differential, we must have

$$\frac{dP}{dy}=\frac{dQ}{dx};$$

putting for P and Q their values, this equation will become

$$\frac{dMz}{dy}=\frac{dNz}{dx};$$

and, performing the differentiations in the numerators, we have

$$\frac{Mdz}{dy}+\frac{zdM}{dy}=\frac{Ndz}{dx}+\frac{zdN}{dx} \quad \ldots\ldots\ldots \quad (b).$$

(3.) When the common factor z is constant, $\frac{dz}{dy}$ and $\frac{dz}{dx}$ being 0, the equation (b) becomes

$$\frac{dM}{dy}=\frac{dN}{dx};$$

and, consequently, the condition necessary that the equation (a) may be a complete differential is fulfilled. But when z is a function of x and y, the determination of z depends on equation (b); and this equation is more difficult to integrate than the proposed one, which contains only the single differential coefficient $\frac{dy}{dx}$, whilst the equation (b) contains three variables, x, y, z, and the two differential coefficients $\frac{dz}{dx}$ and $\frac{dz}{dy}$.

(4.) If the equation be homogeneous, it is very easy to determine the factor; for let $Mdx+Ndy=0$ be an homogeneous equation which becomes integrable when multiplied by an homogeneous function z of x and y; representing the integral of the equation $Mdx+Ndy=0$ by u, we have

$$zMdx+zNdy=du \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (c)$$

and this equation being homogeneous, we deduce by the principles before laid down for this class of equations

$$zMx+zNy=nu \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (d)$$

If now the dimension of M be represented by m, and that of z by k, the dimension of one of the terms zMx or zNy will be $m+k+1$; this value, therefore, being put in place of n, in the preceding equation, we shall have

$$zMx+zNy=(m+k+1)\,u,$$

and dividing the equation (c) by this, we shall find

$$\frac{Mdx+Ndy}{Mx+Ny}=\frac{du}{u}\cdot\frac{1}{m+k+1}.$$

The second side of this equation is a complete differential, and the first must, therefore, be so also; whence it follows that $\frac{1}{Mx+Ny}$ is the factor proper to render the homogeneous equation $Mdx+Ndy=0$ integrable.

(5.) If the common factor z, which ought to render the proposed equation homogeneous, be a function of x alone, we have $\frac{dz}{dy}=0$, which reduces the equation (b) to

$$\frac{zdM}{dy}=\frac{Ndz}{dx}+z\frac{dN}{dx},$$

whence we deduce

$$\frac{Ndz}{dx}=z\left(\frac{dM}{dy}-\frac{dN}{dx}\right),$$

and consequently,

$$\frac{dz}{z}=\frac{\left(\frac{dM}{dy}-\frac{dN}{dx}\right)}{N}\,dx \quad \ldots\ldots\ldots \quad (e);$$

integrating, therefore, we have

$$\log. z=\int\frac{\left(\frac{dM}{dy}=\frac{dN}{dx}\right)}{N}dx$$

$$=\int\frac{1}{N}\left(\frac{dM}{dy}-\frac{dN}{dx}\right)dx;$$

$$\text{or, } z=\epsilon^{\left(\int\frac{1}{N}\left(\frac{dM}{dy}-\frac{dN}{dx}\right)dx\right)} \quad \ldots\ldots\ldots \quad (f).$$

We have only, therefore, to multiply the proposed equation by this factor z, and it will become a complete differential.

Let the equation, for instance, be

$$ydx - xdy = 0;$$

we obtain

$$\frac{dM}{dy} - \frac{dN}{dx} = 2,$$

which reduces the formula (e) to

$$\int \frac{dz}{z} = \int \frac{2dx}{-x};$$

whence we derive, by integrating,

$$\log. z. = -2 \log. x + \log. C = -\log. x^2 + \log. C = \log. \frac{C}{x^2};$$

and passing to numbers, we find

$$z = \frac{C}{x^2};$$

the expression $\frac{C(ydx - xdy)}{x^2}$ will consequently be a complete differential.

(6.) We may find an infinite number of factors which possess the same property. For let z be a factor which renders the equation $Mzdx + Nzdy = 0$ complete; representing the integral of this equation by u, we shall have

$$Mzdx + Mzdy = du;$$

multiplying the two sides by ϕu, we shall obtain

$$\phi u\,(Mzdx + Nzdy) = \phi u du;$$

and ϕu being arbitrary in its form, we may assume for it any function of u, for instance, $2u^2$, and then $2u^2du$ being a complete differential,

$$2u^2 (Mzdx + Nzdy) = 2u^2du$$

must be so likewise: thus the factor $2zu^2$ is one which renders integrable the equation

$$Mdx + Ndy = 0.$$

And we may have an indefinite number of such values of ϕu.

Orders and Degrees of Differential Equations.

(7.) Differential equations, as they may involve any powers of the variables and their successive differential coefficients, are distinguished, like algebraic equations, by *degrees*, according to the *algebraic dimensions* of the different terms involving the variables and their differentials.

They are also distinguished into *orders* according to the highest order of differentiation which they involve.

When a differential equation is formed by successively differentiating an equation of two variables, it is evident that there can result only a differential equation of the first degree; that is, the several differential coefficients will never appear either singly in any power but the first, or combined as factors in a product.

Whenever therefore we meet with a differential equation of a degree higher than the first, we may be sure it is not one which can be referred to the direct differentiation of any primitive equation.

How such equations may arise will be understood from what was shewn in a former part of the work respecting the elimination of constants by differentiation. It there appeared that we might make any number of constants disappear by as many successive differentiations; and in so doing it was evident that the differential equation would rise to a higher degree than the first.

We shall proceed to shew how these considerations apply to the reverse process of integrating such equations.

Theory of Arbitrary Constants.

(8.) An equation, $V=0$, containing only x, y, and constant quantities, may be regarded as the complete integral of some differential equation, which will be of an order dependent on the number of constants which $V=0$ contains: each of such constants being introduced by a successive integration. And if the differential equation be of the nth order, it follows that $V=0$, which is supposed to result from these integrations, must contain at least n arbitrary constants more than the differential equation.

(9.) Let us now take

$$F(x,y)=0, F\left(x, y, \frac{dy}{dx}\right)=0, F\left(x, y, \frac{dy}{dx}, \frac{d^2y}{dx^2}\right)=0 \ldots (g)$$

as the primitive of a differential equation of the second order, and those immediately derived from it; between the two first of these three equations, we may eliminate successively the constants a and b, and so obtain

$$\phi\left(x, y, \frac{dy}{dx}, b\right)=0, \phi\left(x, y, \frac{dy}{dx}, a\right)=0 \ldots . (h).$$

If, without knowing the equation $F(x, y) = 0$, we had arrived at these equations, we should only have to eliminate $\frac{dy}{dx}$ between them in order to obtain $F(x,y) = 0$, which would be the complete integral, since it contains the arbitrary constants a and b.

(10.) If, on the other hand, we eliminated these two constants between the three equations (g), we should arrive at an equation, which, containing the same differential coefficients, might be represented by

$$\phi\left(x, y, \frac{dy}{dx}, \frac{d^2y}{dx^2}\right) = 0 \quad . \quad . \quad . \quad . \quad . \quad (k),$$

but to this result either of the equations (h) would also lead us. For the constant contained in one of these equations and its immediate differential being eliminated between them, we should obtain separately two equations of the second order; and these could not differ from each other or from the equation (k), for if so the values of x and y would not be the same in the two. It follows, therefore, that a differential equation of the second order may arise from two differential equations of the first order, which are necessarily different, since the arbitrary constant in the one is not the same with the arbitrary constant in the other. The equations (h) are called the first integrals of the equation (k), and the equation $F(x, y) = 0$, its second integral.

Let us take, for example, $y = ax + b$, which, on account of its two constants, may be considered as the primitive of an equation of the second order.

We deduce from it by differentiation, and the consequent elimination of a,

$$\frac{dy}{dx}=a,\ y=x\frac{dy}{dx}+b;$$

and these two first integrals of the equation of the second order which we are seeking, being each differentiated in turn, lead us equally, by the elimination of a and b, to the same equation $\frac{d^2y}{dx^2}=0$.

In the case in which the number of the constants is greater than that of the arbitrary constants required, the additional constants, being connected by the same equations, do not introduce any new relation. Let us investigate, for example, the equation of the second order, the primitive of which is

$$y=\tfrac{1}{2}ax^2+bx+c=0.$$

Differentiating this, we obtain

$$\frac{dy}{dx}=ax+b,$$

c and b being then eliminated successively between these equations, we have the two first integrals

$$\frac{dy}{dx}=ax+b,\ y=x\frac{dy}{dx}-\tfrac{1}{2}ax^2+c\ \ldots\ldots\ (l);$$

and combining each of these with their immediate differentials, we arrive, by two different ways, at the same result $\frac{d^2y}{dx^2}=a$. If, on the other hand, we had eliminated the third constant a between the primitive equation and its immediate differential, the result would have been the same; for we should have arrived first at the result that would be furnished us by the elimination of a between the equations (l), and found then $x\frac{d^2y}{dx^2}=\frac{dy}{dx}-b$, an equation which is re-

duced to $\frac{d^2y}{dx^2}=a$, by combining it with the first of the equations (l).

(11.) Applying the same considerations to the differential equation of the third order, if we differentiate the equation $F(x, y)=0$ three times in order, we shall have

$$F\left(x, y, \frac{dy}{dx}\right)=0, \; F\left(x, y, dx, \frac{d^2y}{dx^2}\right)=0,$$

$$F\left(x, y, \frac{dy}{dx}, \frac{d^2y}{dx^2}, \frac{d^3y}{dx^3}\right)=0;$$

and these equations admitting the same values for each of the arbitrary constants which $F(x, y)=0$ contains, we may in general eliminate these constants between this last and the three preceding equations, and so obtain a result which we will represent by

$$f\left(x, y, \frac{dy}{dx}, \frac{d^2y}{dx^2}, \frac{d^3y}{dx^3}\right)=0 \; . \; . \; . \; . \; (m).$$

This equation, from which the three arbitrary constants are eliminated, will be the differential equation of the third order of $F(x, y)=0$; and conversely, $F(x, y)=0$ will be the third integral of the equation (m).

(12.) If we eliminate each of the arbitrary constants successively between the equation $F(x, y)=0$ and that immediately deduced from it by differentiation, we shall obtain three equations of the first order, which will be the second integrals of the equation (m).

Lastly, If we eliminate two of the three arbitrary constants, by means of the equation $F(x, y)=0$, and the equations deduced from it by two successive dif-

ferentiations; i.e. if we eliminate the constants between the equations

$$F(x,y)=0, F\left(x, y, \frac{dy}{dx}\right)=0, F\left(x, y, \frac{dy}{dx}, \frac{d^2y}{dx^2}\right)=0 \ldots (n),$$

there will remain, in the equations arising from the elimination, one of the three arbitrary constants successively; and consequently we shall have as many equations as there are arbitrary constants.

Let a, b, c be these arbitrary constants; the equations, considered only in respect to the arbitrary constants they contain, may be represented thus:

$$\phi c=0, \ \phi b=0, \ \phi a=0 \ . \ . \ . \ . \ (p),$$

and since the equations (n) contribute each of them to the elimination which gives us one of the last, it follows that the equations (p) will be each of the second order; they are called the first integrals of the equation (m).

(13.) Generally, a differential equation of an order n will have a number n of first integrals, which will consequently contain the differential coefficients from $\frac{dy}{dx}$ up to $\frac{d^{n-1}y}{dx^{n-1}}$ inclusively; i.e. a number $n-1$ of differential coefficients; and we see that when these equations are all known, we have only to eliminate the differential coefficients between them to obtain the primitive equation.

On the Particular Solutions of Differential Equations of the First Order.

(14.) We have before seen, on the principle of integrating between certain limits, that a particular integral may always be deduced from the complete integral, by giving a suitable value to the arbitrary constant contained in the latter; this applies also to differential

equations. Suppose, for instance, that we have given the equation

$$xdx + ydy = dy\sqrt{x^2 + y^2 - a^2},$$

the complete integral of which is easily found to be

$$y + c = \sqrt{x^2 + y^2 - a^2}:$$

if, for greater convenience of operating, we clear it of the roots, the proposed expression will become, after dividing by dx and squaring,

$$(a^2 - x^2)\frac{dy^2}{dx^2} + 2xy\frac{dy}{dx} + x^2 = 0 \quad . \; . \; . \quad (q),$$

and we shall have for the complete integral

$$2cy + c^2 - x^2 + a^2 = 0 \quad . \; . \; . \quad (r);$$

when it is evident that by assuming for c a constant arbitrary value $c = 2a$, we shall obtain the particular integral

$$2cy + 5a^2 - x^2 = 0,$$

which will possess the property of satisfying the proposed equation (q) equally well with the complete integral:

For we deduce from this particular integral

$$y = \frac{x^2 - 5a^2}{2c}, \quad \frac{dy}{dx} = \frac{x}{c};$$

by which values the proposed equation is reduced to

$$(x^2 - a^2)\frac{x^2}{c^2} = \frac{x^2}{c^2}(x^2 + c^2 - 5a^2),$$

and this is satisfied by substituting on the second side the value of c^2, which is given by the relation $c = 2a$, established between the constants.

For a long time it was supposed that this property of the complete integral was general, and that when a differential equation between x and y was given, we

could not meet with a finite equation between the same variables, which was not a particular case of the complete integral, by giving, as we have just done, an arbitrary value to the constant; but it was at length discovered that this was not always the case, and Euler himself, in a memoir published in 1756, regarded as a paradox the singular fact that the equation

$$x^2+y^2=a^2 \quad . \; . \; . \quad (s),$$

possesses the property of satisfying the differential equation (q), and yet is not comprised in the complete integral. For the equation (s) being differentiated gives $xdx = -ydy$, and this value and that of x^2+y^2 being substituted in the equation (q) cause all the terms to disappear, and consequently satisfy the equation; nevertheless the equation (s) is not comprised in the complete integral; for whatever be the constant value we give to c in the equation (r), that equation can never lead to the equation (s).

This equation (s), *which satisfies the one proposed, without being contained in the complete integral, is called a particular or singular solution of the equation proposed.* Clairault, about the year 1734, had remarked this fact, and it was for a long time supposed that equations of this sort were not connected with the complete integral; Lagrange shewed that they were dependent on it, and on this subject laid down the theory of which we shall proceed to give a brief account.

(15.) Let $Mdx+Ndy=0$ be a differential equation of the first order of a function of two variables, x and y; this equation may be conceived as arising from the elimination of some constant c between an equation of the same order, which we will represent by $mdx+ndy=0$, and the complete integral $F(x,y,c)=0$,

which we will designate by u. But since all that is required is to take the constant c, so that the equation $Mdx+Ndy=0$ may be the result of the elimination, we see that, provided only the equation $Mdx+Ndy=0$ be satisfied, we may make the constant c itself vary; in which case the complete integral $F(x, y, c)=0$ will possess a still greater degree of generality.

(16.) Suppose, therefore, that the complete integral being differentiated, considering c as variable, we have obtained

$$dy=\frac{dy}{dx}dx+\frac{dy}{dc}dc \quad \ldots\ldots\ldots \quad (t).$$

For greater simplicity we will write this equation thus:

$$dy=pdx+qdc \quad \ldots\ldots\ldots\ldots \quad (v);$$

and it is evident that if, whilst p continues finite, qdc become 0, the result of the elimination of c, considered as variable, between $F(x, y, c)=0$ and the equation (v) will be the same with that of the elimination of c, considered as constant, between $F(x, y, c)=0$ and the equation $dy=pdx$. But that qdc may be $=0$, we must have one of the factors of this equation 0, i. e. we must have

$$dc=0, \text{ or } q=0.$$

In the first of these cases, $dc=0$ gives $c=constant$, which is what takes place for a particular integral; and it will therefore be the second case only that can answer to a particular solution. But q being the coefficient of dc in the equation (t), we see that $q=0$ gives

$$\frac{dy}{dc}=0 \quad \ldots\ldots\ldots\ldots\ldots \quad (w);$$

and this equation may contain c or be independant of

it; if it contain c, two cases may happen: the equation $q=0$ will either contain c along with constants, or will contain c along with the variables. In the first case, the equation $q=0$ will still give c = *constant;* but in the second it will give $c=f(x, y)$, and this value being substituted in the equation $F(x, y, c)=0$, will change it into another function of x and y, which will satisfy the equation proposed without being comprised in its complete integral, and will consequently be a particular solution: we shall, however, have only a particular integral if the equation $c=f(x, y)$, by means of the complete integral, be reduced to a constant.

(17.) When the factor $q=0$ of the equation $qdc=0$ does not contain the arbitrary constant c, we shall know whether the equation $q=0$ gives rise to a particular solution by combining it with the complete integral. For example, if from $q=0$ we deduce $x=M$, and substitute this value in the complete integral $F(x, y, c)=0$, we shall obtain

$$c=\textit{constant}=B, \text{ or } c=fy.$$

In the first case, $q=0$ gives a particular integral; for, changing c into B in the complete integral, we shall merely be giving a particular value to the constant, just as we do when we pass from the complete integral to a particular one. In the second case, on the contrary, the value fy, introduced in place of c in the complete integral, will establish between x and y a relation different from what would result, were we to replace c merely by some constant arbitrary value; and in this case, therefore, we shall have a particular solution. What we have said of y, will apply in like manner to x.

(18.) It happens sometimes that the value of c presents itself under the form $\frac{0}{0}$: this indicates a factor common to the equations u and U, which is foreign to them, and must be made to disappear. But on this point the limits of an elementary work prevent us from entering further.

(19.) We will now apply this theory to the investigation of particular solutions, when the complete integral is given.

Let the equation be

$$ydx - xdy = a\sqrt{dx^2 + dy^2} \quad \ldots\ldots\ldots \quad (\alpha),$$

the complete integral of which is determined as follows:

First, dividing the equation by dx, and writing $\frac{dy}{dx} = p$, we obtain

$$y - px = a\sqrt{1 + p^2} \quad \ldots\ldots\ldots\ldots \quad (\beta);$$

then differentiating in respect of x, y, and p, we have

$$dy - pdx - xdp = \frac{apdp}{\sqrt{1 + p^2}};$$

and observing that $dy = pdx$, this equation is reduced to

$$xdp + \frac{apdp}{\sqrt{1 + p^2}} = 0;$$

which is satisfied by making $dp = 0$.

This hypothesis gives us $p = constant = c$, a value which, being substituted in the equation (β), gives us

$$y - cx = a\sqrt{1 + c^2} \quad \ldots\ldots\ldots\ldots \quad (\gamma);$$

and this equation containing an arbitrary constant c, which does not appear in the proposed equation (α), is consequently the complete integral.

(20.) This being premised, the part qdc of the equation (v) will be obtained by differentiating the equation (γ), considering c as the only variable; which will give

$$xdc+\frac{acdc}{\sqrt{1+c^2}}=0;$$

and consequently from the coefficient of dc, equated to zero, we have

$$x=-\frac{ac}{\sqrt{1+c^2}} \quad \ldots\ldots\ldots\ldots\ldots \quad (\delta).$$

To disengage the value of c; by squaring this equation we find

$$(1+c^2)\,x^2=a^2c^2;$$

whence we deduce

$$c^2=\frac{x^2}{a^2-x^2},\ 1+c^2=\frac{a^2}{a^2-x^2},\ \sqrt{1+c^2}=\frac{a}{\sqrt{a^2-x^2}};$$

and, by means of this last equation, eliminating the root from the equation (δ), we then obtain

$$c=-\frac{x}{\sqrt{a^2-x^2}} \quad \ldots\ldots\ldots\ldots\ldots \quad (\epsilon).$$

This value, and that of $\sqrt{1+c^2}$, being substituted in the equation (γ), we shall have

$$y+\frac{x^2}{\sqrt{a^2-x^2}}=\frac{a^2}{\sqrt{a^2-x^2}};$$

and thence

$$y=\frac{a^2-x^2}{\sqrt{a^2-x^2}}=\sqrt{a^2-x^2},$$

an equation which, being squared, will give

$$y^2=a^2-x^2;$$

and we see that this equation is really a particular solution; for, by differentiating it, we obtain $dy = -\frac{xdx}{y}$; and this value, and that of $\sqrt{x^2+y^2}$, being substituted in the equation (α), reduce it to $a^2 = a^2$.

(21.) In the application which we have just given of the principles demonstrated, art. (16), we have determined the value of c by equating the differential coefficient $\frac{dy}{dc}$ to zero. This process will sometimes prove insufficient; for the equation

$$dy = pdx + qdc$$

being put under the form,

$$Adx + Bdy + Cdc = 0,$$

where A, B, and C, are functions of x and y, we deduce from it

$$dy = -\frac{A}{B}\,dx - \frac{C}{B}\,dc \quad \ldots\ldots\ldots \quad (\zeta)$$

$$dx = -\frac{B}{A}dy - \frac{C}{A}\,dc \quad \ldots\ldots\ldots \quad (\eta);$$

and we see that if all we have said of y, considered as a function of x, be applied to x, considered as a function of y, the value of the coefficient of dc will not necessarily result the same, since it is only requisite that some factor of B should destroy in C a factor different to what a factor of A could destroy in it, in order that the values of the coefficient of dc, on the two hypotheses, may result entirely different. Thus, though very generally the equations $\frac{C}{B} = 0$, and $\frac{C}{A} = 0$, give the same value for c, this does not always happen; and, on this account, when we have determined

c by means of the equation $\frac{dy}{dc}=0$, it will not be altogether useless to examine whether the hypothesis $\frac{dx}{dc}=0$ produces the same result.

(22.) Clairaut first remarked a general class of equations which admit of particular solutions: they are comprised under the form,

$$y=\frac{dy}{dx}x+F\left(\frac{dy}{dx}\right),$$

an equation which we may represent by

$$y=px+Fp \quad \ldots\ldots\ldots\ldots\ldots \quad (\theta);$$

and differentiating, we find

$$dy=pdx+xdp+\frac{dFp}{dp}dp;$$

since $dy=pdx$, this equation is reduced to

$$xdp+\frac{dFp}{dp}dp=0;$$

and dp being a common factor, it may be written thus:

$$\left(x+\frac{dFp}{dp}\right)dp=0.$$

This equation will be satisfied by making $dp=0$, which gives $p=\text{constant}=c$; and, consequently, substituting this value in the equation (θ), we shall find

$$y=cx+Fc;$$

which equation will be the complete integral of the one proposed, since an arbitrary constant c has been introduced by the integration.

If we differentiate this equation in respect to c, we shall have

$$\left(x+\frac{dFc}{dc}\right)\,dc\,;$$

and, consequently, by equating the coefficient of dc to zero, we have the equation

$$x+\frac{dFc}{dc}=0,$$

which, by the substitution of c in the complete integral, will give the particular solution.

We have here only been able to give a very general account of this subject. To enter more fully into it would carry us beyond our limits: the student who is desirous to pursue it must have recourse to larger works. We proceed to an equally brief account of one or two other parts of the subject.

Integration of Differential Equations of the Second Order.

(23.) Differential equations of the second order between two variables may be represented in general by

$$f\left(x,y,\frac{dy}{dx},\frac{d^2y}{dx^2}\right)=0 \quad \ldots\ldots\ldots\ldots \quad (\kappa).$$

We shall not enter upon the integration of such equations in their general form, but merely proceed to examine how the integral can be found in certain particular cases.

(24.) We will first suppose the case where the equation is

$$f\left(x,\frac{dy}{dx},\frac{d^2y}{dx^2}\right)=0 \quad \ldots\ldots\ldots\ldots \quad (\lambda);$$

which, by writing $\dfrac{dy}{dx}=p, \dfrac{d^2y}{dx^2}=\dfrac{dp}{dx},$

will be reduced to

$$f\left(x, p, \frac{dp}{dx}\right) \quad \ldots\ldots\ldots\ldots \quad (\mu).$$

If this equation can be integrated, and we deduce from it $p=X$, we shall readily obtain the value of y; for since the equation $\frac{dy}{dx}=p$ gives us $y=\int pdx$, if we substitute in this equation the value of p, we shall have $y=\int Xdx$. But if the equation (μ), instead of giving us the value of p in terms of x, should give that of x in a function of p, so that we had $x=P$, then integrating $dy=pdx$ by the method of parts, we should have

$$y=px-\int xdp,$$

and substituting in this equation the value of x, we should find

$$y=px-\int Pdp.$$

(25.) Next let us consider the case in which we have

$$f\left(y, \frac{dy}{dx}, \frac{d^2y}{dx^2}\right)=0 \quad \ldots\ldots\ldots\ldots \quad (\nu);$$

making $\frac{dy}{dx}=p$, we shall find $\frac{d^2y}{dx^2}=\frac{dp}{dx}$; and replacing dx by its value $\frac{dy}{p}$, this equation will become

$$\frac{d^2y}{dx^2}=\frac{pdp}{dy}.$$

Writing these values of $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$ in the equation (ν), it is transformed into

$$f(y, p, dy, dp)=0;$$

and if this equation give $p=Y$, we must substitute

this value in the equation $dx=\frac{dy}{p}$, when we shall obtain, by integrating,

$$x=\int\frac{dy}{Y}.$$

If, on the contrary, y results as a function of p, and we have, consequently, $y=P$; to obtain x, we must integrate the equation $dx=\frac{dy}{p}$ by parts, when we shall have

$$x=\frac{y}{p}+\int y\frac{dp}{p^2};$$

and substituting in this equation the value of y, we shall find

$$x=\frac{y}{p}+\int P\frac{dp}{p^2}:$$

having integrated, we must then eliminate P by means of the equation $y=P$.

(26.) When the equation (κ) contains, along with $\frac{d^2y}{dx^2}$, only one of the three quantities $\frac{dy}{dx}$, x, and y, we have, in the first case,

$$f\left(\frac{dy}{dx},\frac{d^2y}{dx^2}\right)=0 \quad \ldots\ldots\ldots\ldots \quad (\pi);$$

and making $\frac{dy}{dx}=p$, and consequently $\frac{d^2y}{dx^2}=\frac{dp}{dx}$, and substituting these values in the equation (π), it will become

$$f\left(p,\frac{dp}{dx}\right)=0.$$

From this equation we deduce

$$\frac{dp}{dx}=P \quad \ldots\ldots\ldots\ldots\ldots \quad (\rho),$$

and consequently

$$x=\int\frac{dp}{P} \quad \ldots\ldots\ldots\ldots \quad (\sigma).$$

On the other hand, the equation $\frac{dy}{dx}=p$ gives us

$$y=\int pdx;$$

and substituting in it the value of dx, given by the equation (ρ), we obtain

$$y=\int\frac{pdp}{P} \quad \ldots\ldots\ldots\ldots \quad (\tau).$$

Having integrated the equations (σ) and (τ), we must eliminate between them the quantity p, to obtain an equation between x and y.

(27.) In the case in which $\frac{d^2y}{dx^2}$ appears combined only with a function of x, we have

$$\frac{d^2y}{dx^2}=X;$$

multiplying by dx, and integrating, we find

$$\frac{dy}{dx}=\int Xdx+C;$$

representing $\int Xdx$ by X_2, we have

$$\frac{dy}{dx}=X_2+C;$$

and multiplying again by dx, and integrating, we obtain

$$y=\int X_2dx+C.$$

(28.) Lastly, when $\frac{d^2y}{dx^2}$ is given in a function of y alone, we have only to integrate the equation

$$\frac{d^2y}{dx^2} = Y.$$

To do this, we multiply the equation by $2dy$, which gives

$$2\frac{dy}{dx} \cdot \frac{d^2y}{dx} = 2Ydy;$$

and the first side being of a form similar to the differential of x^2, we find, by integrating,

$$\frac{dy^2}{dx^2} = \int 2Ydy + C;$$

extracting the square root of this, we obtain

$$\frac{dy}{dx} = \sqrt{\int 2Ydy + C},$$

and, by a second integration, we deduce

$$x = \int \frac{dy}{\sqrt{c + 2\int Ydy}} + C_2.$$

Partial Differential Equations of the First Order.

(29.) An equation which subsists between differential coefficients, combined, according to the case, with variables and constants, is generally a partial differential equation, or contains differential coefficients which indicate that the differentiation is only effected in respect to one of the variables. For greater simplicity, we will first suppose that the proposed function contains two variables, and consider the partial differential equations of the first order, i. e. those which contain only one or more differential coefficients of the first order.

(30.) We will commence with the following equation,

$$\frac{dz}{dx} = a.$$

If z, instead of being a function of two variables x and y, should contain only x, this would be no more than an ordinary differential equation, which, being integrated, would give $z = ax + c$; but since, in the present case, z is by hypothesis a function of x and y, the terms involving y in the function z must have disappeared by the differentiation, since in differentiating in respect of x, y has been considered constant. We must, therefore, in integrating, adhere to the same hypothesis, and suppose that the arbitrary constant is in general a function of y; whence, consequently, we shall have for the integral of the equation proposed,

$$z = ax + \phi y.$$

(31.) If we have also the partial differential equation

$$\frac{dz}{dx} = X,$$

in which X is a function of x; multiplying each side by dx, and integrating, we shall find

$$z = \int X dx + \phi y.$$

(32.) For example, if the function represented by X be $x^2 + a^2$, the integral would be

$$z = \frac{x^3}{3} + a^2 x + \phi y.$$

(33.) In a similar manner to integrate

$$\frac{dz}{dx} = Y,$$

we find

$$z = Yx + \phi y.$$

(34.) We may in like manner integrate every equation in which $\frac{dz}{dx}$ is equal to a function of two variables x and y.

If we have, for example,

$$\frac{dz}{dx}=\frac{x}{\sqrt{ay+x^2}},$$

considering y as constant, we multiply by dx, and integrate according to Int. Calc. (form 8); when representing by ϕy the constant which ought to be added to the integral, we have

$$z=\sqrt{ay+x^2+\phi y}.$$

(35.) Lastly, if we have to integrate the equation

$$\frac{dz}{dx}=\frac{1}{\sqrt{y^2-x^2}},$$

y as before being considered constant, we shall have

$$z=\sin.^{-1}.\frac{x}{y}+\phi y.$$

(36.) Generally, to integrate the equation

$$\frac{dz}{dx}dx=F(x,y)\,dx,$$

we must take the integral in respect of x, and adding a constant function of y to complete it, we shall find

$$z=\int F(x,y)\,dx+\phi y.$$

(37.) From what has been said, we see that, excepting the hypothesis of one of the variables being constant, and the introduction, in the integral, of a constant function of that variable, we follow the same process as in the integration of ordinary differential equations.

(38.) Let us consider now the partial differential equations which contain two differential coefficients of the first order, and let the equation be

$$M\frac{dz}{dx}+N\frac{dz}{dy}=0,$$

in which M and N represent given functions of x and y; we deduce from it

$$\frac{dz}{dy}=-\frac{M}{N}\frac{dz}{dx};$$

and substituting this value in the formula

$$dz=\frac{dz}{dx}dx+\frac{dz}{dy}dy \quad \ldots\ldots\ldots \quad (a);$$

which expresses only the condition that z is a function of x and y, we obtain

$$dz=\frac{dz}{dx}\left(dx-\frac{M}{N}dy\right),$$

or $$dz=\frac{dz}{dx}\cdot\frac{Ndx-Mdy}{N}.$$

Let λ be the factor proper to render $Ndx-Mdy$ a complete differential ds; we shall have then

$$\lambda\,(Ndx-Mdy)=ds \quad \ldots\ldots\ldots \quad (b);$$

and, by means of this equation, eliminating $Ndx-Mdy$ from the preceding one, we obtain

$$dz=\frac{1}{\lambda N}\frac{dz}{dx}\,ds.$$

Lastly, observing that the value of $\frac{dz}{dx}$ is indeterminate, we may assume it such that $\frac{1}{\lambda N}\cdot\frac{dz}{dx}ds$ shall be immediately integrable, which requires that $\frac{1}{\lambda N}\frac{dz}{dx}$ be a function of s; for we know that the differential of every given function of s must be of the form $Fs\,.\,ds$. From this therefore it follows that we must have

$$\frac{1}{\lambda N}\frac{dz}{dx}=Fs,$$

which will change the preceding equation into

$$dz=Fsds;$$

whence we deduce

$$z=\phi s \quad \ldots\ldots\ldots\ldots \quad (c).$$

(39.) If we integrate by this method the equation

$$x\frac{dz}{dy}-y\frac{dz}{dx}=0 \quad \ldots\ldots\ldots\ldots \quad (d),$$

we have in this case $M=-y$, $N=x$, and the equation (a) will consequently become

$$ds=\lambda\ (xdx+ydy).$$

It is evident that the factor proper to render the second side of this equation integrable is 2; substituting, therefore, this value for λ, and integrating, we have

$$s=x^2+y^2;$$

whence, putting this value in the equation (c), we have for the integral of the equation (d)

$$z=\phi\,(x^2+y^2).$$

(40.) Let now the equation be

$$P\frac{dz}{dx}+Q\frac{dz}{dy}+R=0 \quad \ldots\ldots\ldots \quad (e),$$

in which P, Q, R, are functions of the variables x, y, z; dividing by P, and making $\frac{Q}{P}=M$, $\frac{R}{P}=N$, we may put it under the form

$$\frac{dz}{dx}+M\frac{dz}{dy}+N=0 \quad \ldots\ldots\ldots \quad (f);$$

and writing $\frac{dz}{dx}=p$, $\frac{dz}{dy}=q$, this will become

$$p+Mq+N=0 \quad \ldots\ldots\ldots\ldots \quad (g).$$

This equation establishes a relation between the coefficients p and q in the general formula

$$dz = pdx + qdy \quad \ldots\ldots\ldots\ldots \quad (h);$$

without this relation p and q would be entirely arbitrary in the formula, since, as we have already observed, it does no more than indicate that z is a function of x and y, and that function may be any whatever. Thus in the equation (h) p and q must be considered as two indeterminate quantities; and eliminating p by means of the equation (g), we shall obtain

$$dz + Ndx = q\,(dy - Mdx) \quad \ldots\ldots\ldots \quad (k)$$

in which q will still remain indeterminate: but we know that when an equation of this sort holds good, whatever be the value of y, we must have separately

$$dz + Ndx = 0,\ dy - Mdx = 0 \quad \ldots\ldots \quad (l).$$

(41.) If P, Q, and R do not contain the variable z, it will be the same with M and N; in which case the second of the equations (l) will be an equation between the two variables x and y, and may become a complete differential by means of a factor which we will represent by λ, whence we shall have

$$\lambda\,(dy - Mdx) = 0 \quad \ldots\ldots\ldots\ldots \quad (m);$$

and the integral of this equation will be a function of x and y, to which we must add an arbitrary constant s, so that we shall have

$$F\,(x, y) = s,$$

and consequently

$$y = f\,(x, s).$$

This value of y is that given by the second of the equations (l); and in order, therefore, that the two may hold good simultaneously, this value of y must be

substituted in the first of the equations (l); for though the variable y does not explicitly appear in that equation, we see that it may be contained in N.

This substitution, from the nature of the value which we have just found for y, comes to the same thing with considering y, in the first of the equations (l), as a function of x and s; and the first equation being, therefore, integrated on this hypothesis, we shall find

$$z = -\int N dx + \phi s.$$

(42.) To give an example of this mode of integration, let us take the equation

$$x \frac{dz}{dx} + y \frac{dz}{dy} = a\sqrt{x^2 + y^2};$$

comparing it with the equation (f), we have

$$M = \frac{y}{x},\ N = -a\frac{\sqrt{x^2 + y^2}}{x} \quad \ldots\ldots \quad (n),$$

and these values being substituted in the equations (l), they will be changed into

$$dz - a\frac{\sqrt{x^2 + y^2}}{x}dx = 0,\ dy - \frac{y}{x}dx = 0 \quad \ldots \quad (p).$$

Let λ be the factor which renders this last equation integrable; we shall have, then,

$$\lambda\left(dy - \frac{y}{x}dx\right) = 0,$$

or

$$\lambda\left(\frac{xdy - ydx}{x}\right) = 0,$$

and this equation will be integrable, if we make $\lambda = \frac{1}{x}$ since in that case its first side becomes a complete differential (p. 154.)

Equating, therefore, the integral of this equation to an arbitrary constant s, we have

$$\frac{y}{x}=s,$$

and consequently

$$y=sx.$$

By means of this value of y, the first of the equations (p) is changed into

$$dz-a\frac{\sqrt{x^2+s^2x^2}}{x}dx=0,$$

or

$$dz=adx\sqrt{1+s^2};$$

whence, integrating and considering s as constant, we shall obtain

$$z=a\int dx\sqrt{1+s^2}+\phi s,$$

and consequently

$$z=ax\sqrt{1+s^2}+\phi s.$$

Replacing the value of s, there results, lastly,

$$z=ax\sqrt{1+\frac{y^2}{x^2}}+\phi\frac{y}{x},$$

or

$$z=a\sqrt{x^2+y^2}+\phi\frac{y}{x}.$$

(43.) In the most general case, in which the coefficients $P, Q, R,$ of the equation (e) contain the three variables x, y, z, it may happen that the equations (l) contain each of them only the two variables which explicitly shew themselves in the respective equations; and that consequently we may put them under the form

$$dz = f(x, z)dx, \; dy = F(x, y)dx.$$

We cannot integrate these equations independently of each other, by supposing, as in art. (36),

$$z = \int f(x, z)dx + \phi z, \; y = \int F(x, y)dx + \Phi y;$$

for in this case we see that we must assume z to be constant in the first equation, and y to be constant in the second; hypotheses which are contradictory to each other, since one of the three variables, x, y, z, cannot be supposed constant in the first equation without its being also constant in the second.

(44.) The following is the method by which we must integrate the equations (l), in the case in which they contain each of them only the two variables which expressly appear: let μ and λ be the factors which render the equations (l) complete differentials; representing these differentials by dU and dV, we have

$$\lambda(dz + Ndx) = dU, \; \mu(dy - Mdx) = dV,$$

and by means of these values, the equation (k) becomes

$$dU = q\frac{\lambda}{\mu}dV \; . \; . \; . \; (q).$$

Since the first side of this equation is a complete differential, the second must be so likewise, which requires that $q\frac{\lambda}{\mu}$ be a function of V; representing this function by ϕV, the equation (q) becomes

$$dU = \phi V dV;$$

whence we deduce, by integrating,

$$U = \Phi V.$$

(45.) Let us take, for example, the equation

$$xy\frac{dz}{dx}+x^2\frac{dz}{dy}=yz:$$

this being written thus,

$$\frac{dz}{dx}+\frac{x}{y}\frac{dz}{dy}-\frac{z}{x}=0,$$

and compared with the equation (f) we have

$$M=\frac{x}{y},\quad N=-\frac{z}{x},$$

by means of which values the equations (l) become

$$dz-\frac{z}{x}dx=0,\; dy-\frac{x}{y}dx=0,$$

and getting rid of the denominators, we have

$$xdz-zdx=0,\; ydy-xdx=0.$$

The factors proper to render these equations integrable are $\frac{1}{x^2}$ and 2; substituting these, and integrating, we find $\frac{z}{x}$ and y^2-x^2 for the integrals; and putting these values in place of U and V, in the equation $U=\Phi V$, we shall obtain, for the integral of the equation proposed,

$$\frac{z}{x}=\Phi(y^2-x^2).$$

(46.) It is to be observed, that if we had eliminated q instead of p, the equations (l) would have been replaced by the following ones:

$$Mdz+Ndy=0,\; dy-Mdx=0 \quad . \; . \; . \quad (r),$$

and since all that we have said of the equations (l) will apply equally to these, it follows that, in the case in which the first of the equations (l) is not integrable, we

are at liberty to replace those equations by the system of the equations (r); i. e. to employ the first of the equations (r) in place of the first of the equations (l), and then see if the integration be possible.

(47.) For example, if we had

$$az\frac{dz}{dx} - zx\frac{dz}{dy} + xy = 0;$$

this equation, divided by az, and compared with the equation (f), would give us

$$M = -\frac{x}{a},\ N = \frac{xy}{az};$$

whence the equations (l) would become

$$dz + \frac{xy}{az}dx = 0,\ dy + \frac{x}{a}dx = 0;$$

and getting rid of the denominators, we should have

$$azdz + xydx = 0,\ ady + xdx = 0 \quad . \ . \ . \quad (s).$$

Now the first of these equations, containing three variables, cannot be immediately integrated; we shall therefore replace it by the first of the equations (r), when we shall have, instead of the equations (s), the following,

$$-\frac{x}{a}dz + \frac{xy}{az}dy = 0,\ ady + xdx = 0;$$

suppressing $\frac{x}{a}$ as a common factor in the first of these equations, and multiplying the one by $2z$, and the other by 2, we shall find

$$-2zdz + 2ydy = 0,\ 2ady + 2xdx = 0,$$

the integrals of which are

$$y^2 - z^2, \text{ and } 2ay + x^2;$$

and substituting these values in place of U and V, we shall have

$$y^2 - z^2 = \phi(2ay + x^2).$$

(48.) It may be observed, that the first of the equations (r) is no other than that which results from the elimination of dx between the equations (l); and this remark might lead to some general considerations involving the application of the same principle, but they would be of too great a length to be introdnced here.

(49.) We shall terminate this imperfect sketch of partial differential equations of the first order, by the solution of the following problem: *An equation which contains an arbitrary function of one or more variables being given, to find the partial differential equation which has produced it.*

Suppose that we have given

$$z = F(x^2 + y^2);$$

by writing

$$x^2 + y^2 = u \quad . \quad . \quad . \quad (t)$$

the equation will become

$$z = Fu;$$

and since the differential of u must, in general, be a function of u, multiplied by du, we may assume

$$dz = \phi u du.$$

If now we take the differential of z, in respect of x only, i. e. considering y as constant, we must take the differential of u also on the same hypothesis; and, consequently, dividing the preceding equation by dx, we have

$$\frac{dz}{dx} = \phi u \frac{du}{dx} \quad . \quad . \quad . \quad (v);$$

considering, then, x as constant, and y as variable, we find, by a similar process,

$$\frac{dz}{dy}=\phi u\,\frac{du}{dy}\quad . \; . \; . \quad (w).$$

The values of the differential coefficients $\frac{du}{dx}$ and $\frac{du}{dy}$ which enter into the equations (v) and (w), will be obtained by differentiating the equation (t), in respect to x and y successively, which will give

$$\frac{du}{dx}=2x,\ \frac{du}{dy}=2y\,;$$

substituting these values in the equations (v) and (w), we have

$$\frac{dz}{dx}=2x\phi u,\ \frac{dz}{dy}=\ 2y\phi u\,;$$

and eliminating ϕu between these equations, we find, lastly,

$$y\,\frac{dz}{dx}=x\,\frac{dz}{dy}.$$

Partial Differential Equations of the Second Order.

(50.) A partial differential equation of the second order, in which z is a function of two variables x and y, must always contain one or more of the differential coefficients

$$\frac{d^2z}{dx^2},\ \frac{d^2z}{dy^2},\ \frac{d^2z}{dxdy},$$

independently of the differential coefficients of the first order which it may contain.

(51.) We shall confine ourselves to integrating the most simple of the partial differential equations of the second order, and shall commence with

$$\frac{d^2z}{dx^2}=0\,;$$

multiplying this by dx, and integrating in respect of x, we must add to the integral an arbitrary function of y, which gives

$$\frac{dz}{dx}=\phi y;$$

multiplying again by dx, and designating by ψy the function of y to be added to the integral, we find

$$z=x\phi y+\psi y.$$

(52.) Let it be proposed to integrate the equation

$$\frac{d^2z}{dx^2}=P,$$

in which P is a function of x and y; proceeding as before, we shall find first

$$\frac{dz}{dx}=\int Pdx+\phi y;$$

and a second integration will give us

$$z=\int[\int Pdx+\phi y]dx+\psi y.$$

(53.) We might integrate in the same manner

$$\frac{d^2z}{dy^2}=P,$$

and we should find

$$z=\int[\int Pdy+\phi x]dy+\psi x.$$

(54.) The equation

$$\frac{d^2z}{dydx}=P$$

must be integrated, first in respect to one of the variables, and then in respect to the other, which will give

$$z=\int[\int Pdx+\phi y]dy+\phi x$$

(55.) Generally, we may treat in a similar manner any one of the equations

$$\frac{d^n z}{dy^n}=P,\ \frac{d^n z}{dxdy^{n-1}}=Q,\ \frac{d^n z}{dx^2dy^{n-2}}=R,\ \&c.,$$

in which P, Q, R, &c. are functions of x and y; and this will lead to a series of integrations, each of which will introduce an arbitrary function into the integral.

There are several other forms of this kind which admit of easy integration, but upon which the limits of an elementary sketch prevent our entering. The same remark must apply to the omission of several topics of interest and importance; such as the determination of the arbitrary functions which complete the integrals of partial differential equations of the first order; which must be determined by the conditions which belong to the nature of the problems which have produced those equations. For the investigation of these, and other further details of the whole subject, the student must be referred to larger works. He will find many of them discussed in Lacroix's Elementary Treatise: and in the notes to the Cambridge translation some valuable illustrations are given. In Garnier's Leçons de Calcul Integral will be found some important details: but the most complete investigation is that contained in Lacroix's large treatise.

APPENDIX.

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No. I.

COMPRISING

SOME ALTERATIONS

IN THE

"PRINCIPLES OF THE ALGEBRAIC THEORY OF CURVES."

IN the "Principles of the Algebraic Theory of Curves," it has been found that there are one or two points on which difficulties have been experienced by those who were commencing their acquaintance with this branch of mathematics. In the hope of removing those difficulties, the author here subjoins the discussion of one or two of the points alluded to, in what he trusts will be found a better and simplified form.

These paragraphs are so printed that the pages in the original work containing the passages referred to, may be cancelled, and those marked with the same numbers in this Appendix be substituted in their places; viz. pages 23, 33, 37, and 61.

$$\left.\begin{array}{ll}(1.) & \quad\quad Bxy + Cx^2 \\ (2.) & \quad\quad Bxy \\ (3.) & Ay^2 + Bxy \\ (4.) & \pm Ay^2 \quad\quad \mp Cx^2\end{array}\right\} + Dy + Ex + F = 0.$$

(11.) Of these forms however it is essential to observe, that the first supposes $A=0$; in which case the equation cannot be solved for y as above. In this case, however, we might solve it for x, and obtain exactly similar results.

In the second form, where we have both $A=0$ and $C=0$, we can give no solution either for x or y, in the same way as before. It may however be easily shewn that by a mere transference of the locus to other axes, which will not affect its nature, we can obtain an equation which will come under one of the other forms, and therefore shews that the locus still belongs to the same class. Taking therefore the case $A=0$, $C=0$, let us proceed to transfer the locus to new axes X_2 Y_2, with the same origin whose angle of ordination is ω_2, and so placed that X_2 bisects the angle XY, each half of which we will write $=\phi$. Then we have

$$\angle\, xx_2 = -\phi \quad xy_2 = \omega_2 + \phi \quad x_2y = \phi \quad yy_2 = \omega - \phi.$$

Making these substitutions the general formula, (Introd. 5.) gives

$$y = \frac{-x_2 \sin.\phi + y_2 \sin.(\omega_2 + \phi)}{\sin.\omega}$$

$$x = \frac{x_2 \sin.\phi + y_2 \sin.(\omega_2 - \phi)}{\sin.\omega}$$

We have then only to substitute these values in the equation

$$Bxy + Dy + Ex + F = 0,$$

and it becomes (removing the accents)

$$\left.\begin{array}{l}\frac{B}{\sin.^2 2\phi}\left[\left(-x\sin.\phi+y\sin.(\omega+\phi)\right)\left(x\sin.\phi+y\sin.(\omega-\phi)\right)\right] \\ +\frac{D}{\sin.2\phi}\left[x\sin.\phi+y\sin.(\omega+\phi)\right] \\ +\frac{E}{\sin.2\phi}\left[x\sin.\phi+y\sin.(\omega-\phi)\right]+F\end{array}\right\}=0$$

Performing the multiplication in the first term it becomes

$$\frac{B}{\sin.^2 2\phi}\left\{\begin{array}{l}-x^2\sin.^2\phi-xy\sin.\phi\sin.(\omega-\phi) \\ \qquad +xy\sin.\phi\sin.(\omega+\phi) \\ \qquad +y^2\sin.(\omega+\phi)\sin.(\omega-\phi).\end{array}\right\}$$

If we then collect the terms, and arrange them according to the powers of y and x, and write single letters for the coefficients, we shall have evidently an equation of the form

$$\alpha y^2+\beta xy+\gamma x^2+\delta y+\epsilon x+F=0.$$

Thus it appears that the locus of the second of the above forms properly belongs to the class of the hyperbola, being only differently situated with respect to the axes.

(12.) 2dly. The condition which gives the ellipse, or $-(B^2-4AC)$ will remain unaltered only on the supposition $B=0$; for since B^2 is essentially positive, neither A nor C can be $=0$; and when $B=0$, we must have $4AC$ positive, in order that the negative sign may remain: that is, A and C must each have the same sign. Hence the only variation which the equation admits in this case is

$$(5)\quad Ay^2+Cx^2+Dy+Ex+F=0.$$

(13.) 3dly. The condition which gives the parabola $B^2-4AC=0$, will remain unaltered if we have (1) $B=0$ and $A=0$, or (2) $B=0$ and $C=0$.

Hence the variations of the general equation will be,

In the central curves where q has a finite value this general expression may be written

$$y^2 = q\left(x^2 + \frac{p}{q}x\right).$$

Or since (16) we have $\frac{p}{q} = 2a$ this becomes

$$y^2 = q(x + 2a)x.$$

If we proceed to transfer the origin to the *centre*, the new value of x will be in general

$$x_2 = x + a, \text{ whence } x = x_2 - a.$$

Substituting this value, and suppressing the accent, the *equation referring to the centre* will be

$$y^2 = q(x+a)(x-a) = q(x^2 - a^2).$$

Or if we write for q its value $\frac{p}{2a}$ this becomes

$$y^2 = \frac{p}{2a}(x^2 - a^2).$$

Or again, if we write $b^2 = \frac{pa}{2}$, whence $q = \frac{b^2}{a^2}$,

$$y^2 = \frac{b^2}{a^2}(x^2 - a^2).$$

And similarly the *equation referred to the vertex* will be

$$y^2 = \frac{p}{2a}(2ax + x^2);$$

or $$y^2 = \frac{b^2}{a^2}(2ax + x^2).$$

These forms being general for the central curves, if we supply the proper signs, we have $\pm q$, and therefore $\pm 2a$, which will give the several forms (the upper sign belonging to the hyperbola, the lower to the ellipse)

$$y^2 = \pm\frac{p}{2a}(\pm 2ax + x^2) \quad . \quad . \quad . \quad (C)$$

$$y^2 = \pm\frac{p}{2a}(x^2 - a^2) \quad . \quad . \quad . \quad . \quad (D).$$

$$y^2 = \pm\frac{b^2}{a^2}(\pm 2ax + x^2) \quad . \quad . \quad . \quad (c)$$

$$y^2 = \pm\frac{b^2}{a^2}(x^2 - a^2) \quad . \quad . \quad . \quad . \quad (d)$$

Dividing this last form by b^2, we obtain another form which is often used:

$$\frac{y^2}{b^2} \mp \frac{x^2}{a^2} = \mp 1.$$

Or changing all the signs

$$\frac{x^2}{a^2} \mp \frac{y^2}{b^2} = 1.$$

(24.) The general expression $y^2 = px + qx^2$ corresponds to the case of a diameter coinciding with the axis X, and having its ordinates parallel to Y: and in the ellipse its conjugate diameter also parallel to Y. And since the angle of ordination is arbitrary, this may be applied to any pair of conjugate diameters.

In the substitution $q = \frac{b^2}{a^2}$ in form (d), it is evident that b^2 or qa^2 is the value of y^2 when $x = 0$. But in the ellipse in this case y becomes the semi-conjugate diameter: hence the *parameter is a third proportional to the conjugate diameters.* And since this equation applies to the hyperbola also, we may have lines through the centre, parallel to the ordinates of any diameters, and determined in length by this proportion, which may be considered as conjugate diameters.

This equation gives us the property that the *rectangles of the abscissæ*, or in the parabola, *the abscissæ* simply, *are as the squares of the ordinates.* Also in the parabola the parameter is a third proportional to any abscissa and its ordinate.

By these properties we are enabled to *identify* the

$$b^2 \left\{ \begin{matrix} > \\ < \\ = \end{matrix} \right\} a^2m^2 \text{ or } \frac{b}{a} \left\{ \begin{matrix} > \\ < \\ = \end{matrix} \right\} m \quad \begin{matrix} x \text{ and } y \\ \text{will be} \end{matrix} \left\{ \begin{matrix} \text{real} \\ \text{impossible} \\ = \infty . \end{matrix} \right.$$

In the first case the diameter meets each of the opposite curves to which the equation belongs.

In the second, it never meets them.

In the third case, if we have $\frac{b}{a} = m$, the diameter only meets the curve at a point infinitely remote: or in other words, if in the hyperbola the equation of a line through the centre be

$$\alpha y + \beta x = 0,$$

with the condition $\frac{\beta}{\alpha} = \frac{b}{a}$, that line never meets the curve or ceases to be a diameter according to the definition; but it is the limit in position between those diameters which do, and those which do not meet the curve.

(28.) But we may consider these lines in another point of view.

If we take the equation of the hyperbola when the origin is at the centre, and *suppose* x *to become infinite*, the equation

$$y^2 = \frac{b^2}{a^2} (x^2 - a^2),$$

since a vanishes in comparison with x, is reduced to

$$y^2 = \frac{b^2}{a^2} x^2,$$

or since now both sides are complete squares,

$$y = \pm \frac{b}{a} x,$$

an equation of the first degree.

The supposition of x becoming infinite is equivalent

to finding the form of the equation when referring to a part of the locus infinitely remote : and we find that in this case it becomes the equation to two straight lines equally inclined to the principal diameter. With these lines the curve may therefore be considered to be identified at a distance infinitely great : in other words, at very great distances it approaches indefinitely towards such coincidence. And this circumstance of the *equation taking the form of that of a straight line when* x *is made infinite*, affords the *algebraic definition of asymptotes.*

We see immediately from the nature of the equation that this applies only to the hyperbola. In the ellipse x cannot become infinite, and in the parabola the equation $y^2=px$ retains its form for all values of x, and is not reducible to a lower degree.

In the hyperbola it is evident that if we take the equation with rectangular axes, we have

$$\frac{b}{a}=\tan.\ \phi$$

for the angle at which the asymptotes are inclined to the axis X.

(29.) If we take the equation to the hyperbola

$$y^2=\frac{b^2}{a^2}(x^2-a^2),$$

as referring to rectangular axes, and transfer the locus to oblique axes having the same origin, and inclined so that the axis X bisects their angle of ordination, each half of which we will write $=\phi$, then we shall have in the case of the general formula [Introd. 6. case 3.]

$$\angle\, xx_2=-\phi$$

$$xy_2=\phi,$$

or the formula becomes

$$y = -x_2 \sin.\phi + y_2 \sin.\phi$$
$$x = x_2 \cos.\phi + y_2 \cos.\phi,$$

or

$$y = (y_2 - x_2) \sin.\phi$$
$$x = (y_2 + x_2) \cos.\phi.$$

Substituting these values in the equation, it becomes (suppressing the accents)

$$(y-x)^2 \sin.^2\phi = \frac{b^2}{a^2}\Big((y+x)^2 \cos.^2\phi - a^2\Big).$$

Now if we take the asymptotes for these new axes, we have

$$\frac{b^2}{a^2} = \frac{\sin.^2\phi}{\cos.^2\phi},$$

and the equation becomes

$$(y-x)^2 \sin.^2\phi = (y+x)^2 \sin.^2\phi - \frac{a^2 \sin.^2\phi}{\cos.^2\phi}.$$

But on expanding the powers and subtracting, they are reduced to $-4xy$. Hence, multiplying by $\frac{\cos.\phi}{\sin.\phi}$, there results,

$$4xy \sin.\phi \cos.\phi = a^2 \frac{\sin.\phi}{\cos.\phi}$$

$$xy\, 2 \sin.\phi \cos.\phi = \frac{ab}{2}.$$

This, by trigonometry, is equivalent to

$$xy \sin.2\phi = \frac{ab}{2},$$

which is the equation of the hyperbola referred to its asymptotes as axes, or, as it is commonly called, *the equation of the hyperbola between its asymptotes.* This equation shews the property that the areas of

the parellelograms contained by the asymptotes and the two coordinates of any point in the curve, are all equal to one another, and to half the rectangle of the semi-axes.

If 2ϕ be a right angle the equation becomes simply

$$xy = \frac{ab}{2},$$

and the hyperbola is termed rectangular, or equilateral.

In the general investigation of the equation we found, that in the case of two equal roots the locus becomes two straight lines intersecting. In the simplified form of the equation, under the condition $B=0$, $D=0$, and $F=0$, the equations to these lines become

$$y = \pm\sqrt{q}\,(x - x_2)$$
$$= \pm\frac{b}{a}\,(x - x_2).$$

Their point of intersection now lies in the axis X, with which the diameter coincides, and is obviously at the bisection of the line intercepted between the two vertices VV_2 with any other values of the roots.

If we suppose rectangular axes, the coefficient $\frac{b}{a}$ which gives the inclination of the line to the axis X, if ϕ be the angle of inclination, becomes $= \frac{\sin. \phi}{\cos. \phi}$.

Thus these lines coincide with those which would be the asymptotes of the curve if it existed; or the curve in this case, as it were, merges into a coincidence with its asymptotes.

half right angles with the axis at the origin, where they are also tangents to the curve.

(11.) Hence we may derive a geometrical construction.

The equation of the equilateral hyperbola with the origin at the centre is

$$x^2 - y^2 = a^2;$$

that of its tangent, at a point $\alpha\,\beta$,

$$y\beta - x\alpha = -a^2, \quad \text{or, } y - \frac{\alpha}{\beta}x = -\frac{a^2}{\beta};$$

whence the equation to the perpendicular on this tangent through the centre is

$$y + \frac{\beta}{\alpha}x = 0; \text{ whence } \beta = -\frac{\alpha y}{x}.$$

Substituting this value in the equation to the tangent, since for the point of concourse their coordinates are identical, we have

$$-\frac{\alpha y^2}{x} - x\alpha = -a^2;$$

$$\text{whence } \alpha = \frac{a^2 x}{y^2 + x^2} \therefore \beta = \frac{a^2 y}{y^2 + x^2}.$$

Then substituting these values of α and β in the equation to the hyperbola, of which they are coordinates,

$$\frac{a^4 x^2 - a^4 y^2}{(y^2 + x^2)^2} = a^2,$$

$$\text{or } a^2(x^2 - y^2) = (y^2 + x^2)^2;$$

which is the equation to *the locus of the concourse of the tangent and central perpendicular in an equilateral hyperbola;* and is evidently the same as that of Bernouilli's lemniscata.

By a method precisely similar may be investigated the loci of the concourse of the tangent and central perpendicular generally of any hyperbola and of the ellipse, which give a series of analogous curves.

*N*TH DEGREE.

The fifth and higher degrees of equations present no instances of importance: we shall therefore proceed to the general properties of algebraic curves of the nth degree.

SECTION V.

EQUATIONS OF THE *N*TH DEGREE.

(1.) A general and complete equation of the nth degree, is one consisting of terms which involve the variables x and y in all the combinations of the powers of each, such that the highest sum of the exponents shall be $=n$, each term having a constant coefficient. These combinations are easily exhibited; and most clearly, when arranged in the following tabular form [a].

y^n	$y^{n-1}x$	$y^{n-2}x^2$	$y^{n-3}x^3$	—	—	yx^{n-1}	x^n
y^{n-1}	$y^{n-2}x$	$y^{n-3}x^2$	—	—	yx^{n-2}	x^{n-1}	
—	—	—	—	—	—		
—	—	—	—	—			
y^3	y^2x	yx^2	x^3				
y^2	yx	x^2					
y	x						
1							

g De Gua's improvement on a similar arrangement by Newton.

No. II.

ON THE GENERAL THEORY OF ASYMPTOTES.

If in the equation to a curve we make $x=\infty$, and by virtue of this supposition find that from the evanescence of certain terms we have a new equation resulting for y in terms of x, that equation may be understood to be that belonging to a portion of the locus at an infinite distance: or more properly to be that of a curve with which at remote distances the former curve tends to coincide. This second curve is called an asymptote to the first; or if the new equation be of the first degree, we have a rectilinear asymptote. This is the general algebraic definition of an asymptote.

If we take the equation of a curve, and investigate a series expressing the value of y in descending powers of x, which is effected generally by Lagrange's theorem, and confine ourselves to the term involving the first power of x, on the supposition $x = \infty$, we shall have the equation of a rectilinear asymptote. If we take two terms, we shall have a curve which is asymptotic to the first, and approaches it more nearly than the rectilinear asymptote. If we proceed to include three, four, &c. terms, we shall have a series of curves each approaching the first curve more nearly.

This theory will be sufficiently illustrated by the following example:

Let there be given, as the equation of a curve,

$$ax^4 - by^4 + c^3xy = 0.$$

Here, in order to apply Lagrange's theorem, we have, by writing $y^4 = u$,

$$\frac{a}{b}x^4 + \frac{c^3}{b}u^{\frac{1}{4}}x - u = 0;$$

and comparing this with the form

$$z + x\phi u - u = 0,$$

we have $y = fu = u^{\frac{1}{4}}$ $\qquad z = \frac{a}{b} x^4$

$$\phi u = u^{\frac{1}{4}} \qquad (x) = \frac{c^3}{b} x.$$

$$\text{Hence } fz = z^{\frac{1}{4}} = \frac{a^{\frac{1}{4}}}{b^{\frac{1}{4}}} x \qquad \phi z = z^{\frac{1}{4}} = \frac{a^{\frac{1}{4}}}{b^{\frac{1}{4}}} x$$

$$\frac{dfz}{dz} = \frac{1}{4} z^{-\frac{3}{4}} = \frac{1}{4} \frac{a^{-\frac{3}{4}}}{b^{-\frac{3}{4}}} x^{-3} \text{ \&c.}$$

Hence we have the developement

$$y = \frac{a^{\frac{1}{4}}}{b^{\frac{1}{4}}} x + \left(\frac{1}{4} \frac{a^{-\frac{3}{4}}}{b^{-\frac{3}{4}}} . \frac{1}{x^3} . \frac{a^{\frac{1}{4}}}{b^{\frac{1}{4}}} x . \frac{c^3}{b} x \right) + \text{\&c.}$$

Which becomes

$$y = \frac{a^{\frac{1}{4}}}{b^{\frac{1}{4}}} x + \frac{c^3}{4a^{\frac{1}{2}} b^{\frac{1}{2}} x} - \frac{c^6}{4^2 . 1 . 2 . b^{\frac{3}{4}} a^{\frac{5}{4}} x^3} + \text{\&c.}$$

Here therefore

$$y = \frac{a^{\frac{1}{4}}}{b^{\frac{1}{4}}} x,$$

is the equation of the rectilinear asymptote; again,

$$y = \frac{a^{\frac{1}{4}}}{b^{\frac{1}{4}}} x + \frac{c^3}{4a^{\frac{1}{2}} b^{\frac{1}{2}} x}$$

is the equation to a curve, which is an asymptote approaching nearer to the original curve than the rectilinear asymptote.

If we took three terms we should have a curvilinear asymptote of a higher order; and so on for the rest of the series.

This method was given by Stirling, in his Commentary on Newton's enumeration of lines of the third order; and by Lagrange, Theorie des Fonctions, part ii. ch. 2.